University of Illinois Spring 2008

ECE 313: Problem Set 7

Independent Events; System Reliability; Decision-Making

This Problem Set contains six problems

Due:	Wednesday March 12 at the beginning of class.
Reading:	Ross Ch. 3.4; PowerPoint Lectures 15-20
Noncredit Exercises:	**Chapter 3:** Problems 53, 58, 59, 62, 63, 70-74, 78, 81
Reminders:	No class on Friday March 7 on account of EOH

1. An ECE 313 student seeks predictions from three psychics A, B. and C as to whether the student will pass the course. The psychics predict that the student will pass with probabilities $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$, and $P(C) = \frac{3}{4}$ respectively, Assume that A, B, and C are *mutually independent* events, and let D denote the event that at least two of the three psychics predict that the student will pass.

   (a) What is $P(D)$?

   (b) Given that event D occurred, what is the probability that the most pessimistic psychic A predicted that the student will *fail* ECE 313?

2. The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides independently of the other groups whether to support or oppose the motion. All members of the group then vote in accordance with the caucus decision. If you believe that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a real bargain price . . .

   (a) Let A, B, C, and D respectively denote the events that the four groups vote to eliminate all income taxes on capital gains. Suppose that the probabilities of these independent events are $P(A) = 0.9, P(B) = 0.6, P(C) = 0.5$ and $P(D) = 0.2$. What is the probability that the bill passes?

   (b) The President vetoes the bill as a budget-breaker. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities $P(E) = 0.99, P(F) = 0.4, P(G) = 0.6$, and $P(H) = 0.1$, what is the probability that the motion to override the veto passes ?

   Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

3. The ToyAuto Company needs to decide which of the following two methods provides more reliable transportation:

   - a single gigantic car with N engines, N transmissions, N brakes, ... etc. that works (i.e. provides us with transportation) as long as *at least one* of its engines and *at least one* of its transmissions, and *at least one* of its brakes ... works.
   - N separate ordinary cars that fail as soon as any one of their parts fail, but which together provide us with transportation as long as at least one car is in working condition.

   Each car is made of M different types of parts, and (at least) one part of each different type must work for the car to work. Each part fails with probability p and all the failures are independent events.

   (a) For each method, find the probability of system failure (we have no transportation!) in terms of p, N and M

(b) Suppose that $M = 5$ and $p = 0.2$. If it is desired that the system failure probability be less than 0.001, what should N be with each method?

(c) Repeat part (b) assuming that $M = 1000$.

4. A photodetector counts photons for 10 nanoseconds to decide if a distant light source is emitting light. When the light source is *not* emitting light, some photons are still counted by the detector due to the ambient background radiation. The number of photons counted in 10 nanoseconds is modeled as a Poisson random variable $\mathcal{X}$ whose parameter λ has value $\ln(9)$ if the light source *is not* emitting light (hypothesis H_0), and value $\ln(27)$ if the source *is* emitting light (hypothesis H_1). The maximum-likelihood detector decides that H_1 is the true hypothesis if and only if the *likelihood ratio* $\Lambda(u) = p_1(u)/p_0(u)$ exceeds 1.

   (a) What is the value of $\Lambda(k)$ when k photons have been counted?

   (b) What value(s) of $\mathcal{X}$ result in a decision in favor of hypothesis H_1?

   (c) Compute the *false alarm* probability P_{FA} and the *missed detection* or *false dismissal* probability P_{MD} of the maximum-likelihood decision rule.

5. A coin is tossed repeatedly (independent trials) until a Head is observed for the first time. $\mathcal{X}$ denotes the number of trials needed to observe the first Head. The two hypotheses are

   - H_1: $\mathcal{X} \sim \mathsf{Geometric}(p_1)$
   - H_0: $\mathcal{X} \sim \mathsf{Geometric}(p_0)$

   where $0 < p_1 < p_0 < 1$.

   (a) State the maximum-likelihood decision rule as a threshold test on the observed value of $\mathcal{X}$.

   (b) Let π_0 and $\pi_1 = 1 - \pi_0$ respectively denote the *a priori* probabilities of hypotheses H_0 and H_1 and assume that $0 < \pi_0 < 1$.
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_1 regardless of the value of the observation $\mathcal{X}$?
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_0 regardless of the value of the observation $\mathcal{X}$?

6. ["Give me an F!" shouted the cheerleader...] H_0, H_1, and H_2 respectively denote the hypotheses that a student is excellent, good, or average (there are no poor students). The number of grade points earned by the student in a course is a random variable $\mathcal{X}$ that takes on values 3, 6, 9, and 12 only. The professor knows that the pmf of $\mathcal{X}$ when H_0 is true is $p_0(12) = 0.75, p_0(9) = 0.15, p_0(6) = 0.08, p_0(3) = 0.02$, that is, an excellent student has 75% chance of doing well enough on the exam to get an A, 15% chance of a B, etc. Similarly, when H_1 is the true hypothesis, the pmf of $\mathcal{X}$ is $p_1(12) = 0.15, p_1(9) = 0.6, p_1(6) = 0.15, p_1(3) = 0.1$, while if H_2 is true, $p_2(12) = 0.05, p_2(9) = 0.1, p_2(6) = 0.65, p_2(3) = 0.2$. The professor observes $\mathcal{X}$ and must decide which of the hypotheses H_0, H_1, H_2 is true.

   (a) What is the professor's maximum-likelihood decision rule?

   (b) What is the probability that an excellent student is mistakenly labeled as good? What is the probability that an excellent student is mistakenly labeled as average? What is th probability that an average student is classified either as good or as excellent?

   (c) If $P(H_0) = 0.2, P(H_1) = 0.55$, and $P(H_2) = 0.25$, what is the probability that the maximum-likelihood decision rule mis-classifies students?

   (d) What is the Bayes' decision rule corresponding to these probabilities and what is the probability that the Bayes' decision rule mis-classifies students?

   (e) At the Lake Wobegon campus of the University, 95% of students are excellent and 5% are good (and thus they are all above average!) What is Bayes' decision rule in this case? That is, what does the Bayesian professor decide about a student based on the four possible results of the student's exam?