University of Illinois Spring 2008

ECE 313: Problem Set 6

Conditional Probability; Theorem of Total Probability; Bayes' Rule

This Problem Set contains six problems

Due:	Wednesday February 27 at the beginning of class.
Reading:	Ross Ch. 3 except Section 3.4; PowerPoint Lectures 9-14
Noncredit Exercises:	**Chapter 3:** Problems 1, 2, 5, 10, 12, 16, 31, 38, 39, 44
	Theoretical Exercises 1, 2, 8; Self-Test Problems 1-10.
Reminders:	No class on Friday February 29
	Hour Exam I on Monday March 3, 7:00 p.m. – 8:00 p.m
	Rooms 165 and 168 Everitt Laboratory

1. (a) The theorem of total probability states that

$$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c).$$

Show that this result still holds when *everything* is conditioned on event B, that is, prove that

$$P(A|B) = P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B).$$

(b) Bayes' rule states that $P(A|C) = \dfrac{P(C|A)P(A)}{P(C|A)P(A) + P(C|A^c)P(A^c)}$.
Show that this result still holds when *everything* is conditioned on event B, that is, prove that

$$P(A|BC) = \frac{P(C|AB)P(A|B)}{P(C|AB)P(A|B) + P(C|A^cB)P(A^c|B)}.$$

(c) Prove the following result: $P(A|BC) = \dfrac{P(C|AB)P(A|B)}{P(C|B)}$.

2. An urn contains r red and g green balls. Two balls are drawn at random from the urn, with the first ball being returned to the urn (which is then well shaken) before the second ball is drawn. Let R_1 and R_2 respectively denote the events that the first and second balls are red.

(a) What are $P(R_1)$ and $P(R_2)$?

(b) Now suppose that when the first ball is returned to the urn, c *additional* balls of the *same color* are also put into the urn (which is then well shaken before the second ball is drawn.) Clearly $P(R_1)$ is the same as before, but what is $P(R_2)$ now? Remember that the urn now contains $r+g+c$ balls. Simplify your answer and compare to the value of $P(R_2)$ that you obtained in part (a).

(c) For the experiment of part (b), what is the conditional probability that the urn contained $r+c$ red balls given that R_2 occurred?

3. Todd is teaching a probability class and at the end of each week he can either be up-to-date with his lecture notes or he may have fallen behind. If he is up-to-date in a given week, the probability that he will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If he is behind in a given week, the probability that he will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). Let U_i and B_i be the events that Todd is up-to-date or behind, respectively, after i weeks, $i \geq 0$. Todd is (by default) up-to-date at the start of the semester and thus $P(U_0) = 1,\ P(B_0) = 0$.

   (a) Find $P(U_1)$ and $P(B_1)$. Is $P(U_1) + P(B_1) = 1$? Why or why not?

   (b) Find $P(U_2)$ and $P(B_2)$. Is $P(U_2) + P(B_2) = 1$? Why or why not?

   (c) For $i \geq 0$, what are $P(U_{i+1}|U_i)$ and $P(B_{i+1}|B_i)$?

   (d) More generally, express $P(U_{i+1})$ and $P(B_{i+1})$ in terms of $P(U_i)$ and $P(B_i)$. You may find it convenient to express the *column vector* $[P(U_{i+1}), P(B_{i+1})]^T$ in the form $A \cdot [P(U_i), P(B_i)]^T$ where A is a 2×2 matrix.
   Now show that $P(U_i) + P(B_i) = 1$ implies that $P(U_{i+1}) + P(B_{i+1}) = 1$.

   (e) Given that Todd is behind after three weeks, what is the probability that he was behind after two weeks? (*hint: use Bayes' rule*).

   (f) (**Noncredit exercise** for those who have taken Math 386 and/or ECE 410)
   Part (d) shows that the sequence $\{P(U_i)\}$ satisfies a linear difference equation. Solve this equation to get a general expression for $P(U_i)$ as a function of i, and find $\lim_{i\to\infty} P(U_i)$.
   Note: The semester will not last infinitely long; it may just seem that way!

4. A mobile station (MS) is in one of four disjoint cells, numbered 1 through 4. When the MS must be found, it is paged in one cell at a time. Due to channel fading, whenever the MS is paged in the right cell, the page is successful with probability 0.9, and otherwise the page is a miss. The MS is first paged in cell 1. If it is not found there (this happens if the MS wasn't in cell 1, or if it was in cell 1 and the first page was a miss), it is paged in cell 2. If it is not found there, it is paged in cell 3. If it is not found there, it is paged in cell 4. If the MS is not found after all four pages, a second round of pages is started. If the MS hasn't been found after a second round of pages, the overall search is a failure. Let E_i denote the event that the MS is located in cell i, and suppose that $P(E_i) = \frac{5-i}{10}$ for $1 \leq i \leq 4$. Let F_k be the event that the MS is found on the k^{th} page, and let G_k be the event that the MS is not found within the first k pages, for $1 \leq k \leq 8$.

   (a) What is $P(F_1)$?

   (b) What is $P(E_1|F_1)$?

   (c) Find $P(G_8)$, which is the probability that the overall search is a failure.

   (d) Find $P(F_2|G_1)$.

   (e) Find $P(F_4|G_3)$.

   (f) Find $P(F_8|G_7)$.

5. Monty Hall, the host of the TV game show "Let's Make A Deal" shows you three curtains. One curtain conceals a car, while the other two conceal goats. All three curtains are equally likely to conceal the car. He offers you the following "deal": pick

a curtain, and you can have whatever is behind it. When you pick a curtain, instead of giving you your just deserts, Monty (who knows where the car is) opens one of the remaining curtains to show you that there is a goat behind it, and offers the following "new, improved deal" : you can either stick with your original choice, or switch to the remaining (unopened) curtain. Amidst the deafening roars of "Stand pat" and "Switch, you idiot" from the crowd, Monty points out that previously your chances of winning were 1/3. Now, since you know that the car is behind one of the two unopened curtains, your chances of winning have increased to 1/2, and thus the new improved deal is indeed better. Use the theorem of total probability to determine:

(a) the probability of winning if you always switch.

(b) the probability of winning if you never switch.

(c) whether Monty is correct in asserting that if you choose randomly between the two unopened curtains, you have a probability of winning of 1/2.

(d) Having disposed of your goat, you return the next day to the show, and this time, Monty calls you *and* your friend to come on down and choose one curtain each. Which is better: to be the first to pick a curtain or the second? Or does it not make a difference? This time, Monty opens the curtain chosen by your friend to reveal a goat and sends him back. He now asks whether you want to stick with your original choice or switch to the the third (unchosen) curtain. Which choice gives you a larger chance of winning the car?

Note: Everybody knows that the rules of the game of parts (a)-(c) are that Monty always opens one of the two unchosen curtains and he always offers the "new improved deal," i.e. he never opens a curtain to reveal the prize (saying "Oops, you lose; return to your seat."). In the game of part (d), he always opens one of the chosen curtains to eliminate one of the contestants and then always offers the other contestant the chance to switch.

6. Let $\mathcal{X}$ and $\mathcal{Y}$ denote two discrete random variables taking on values $1, 2, 3$. $\mathcal{X}$ denotes a number that we wish to transmit over a channel using one of the three signals s_1, s_2 and s_3. Let $s_{\mathcal{X}}$ denote the signal that is transmitted. Noise in the channel can corrupt the signal, and thus it is possible that the received signal $s_{\mathcal{Y}}$ is not the same as the transmitted signal $s_{\mathcal{X}}$. In particular, the *transition matrix* below gives the (conditional) probability that the receiver hears $s_{\mathcal{Y}}$ when the transmitter sends $s_{\mathcal{X}}$.

Transmitted	Received $\mathcal{Y}$		
$\mathcal{X}$	1	2	3
1	0.8	0.1	0.1
2	0.05	0.9	0.05
3	0.15	0.05	0.8

For example, this table is saying that a transmitted s_1 is received as an s_1, or s_2 or s_3 with probabilities $0.8, 0.1$, and 0.1 respectively.

(a) Suppose that $\mathcal{X}$ has pmf $p_{\mathcal{X}}(1) = 0.5$, $p_{\mathcal{X}}(2) = 0.25$, $p_{\mathcal{X}}(3) = 0.25$. What is the pmf of $\mathcal{Y}$?

(b) Given that the receiver heard $\mathcal{Y} = 3$, what are the *conditional* probabilities of $\{\mathcal{X} = 1\}$? $\{\mathcal{X} = 2\}$? $\{\mathcal{X} = 3\}$?