

ECE 313: Problem Set 4

Discrete Random Variables

This Problem Set contains six problems

Due: Wednesday February 13 at the beginning of class.
Reading: **Chapter 4**
Noncredit Exercises DO NOT turn these in. **Chapter 4:** Problems 2, 7, 13, 28, 35, 39, 40-43
 Theoretical Exercises 11, 13, 15; Self-Test Problems 1-10.

1. If the weather is good (which happens with probability 0.6), Alice walks the 2 miles to class at a speed of $V = 5$ miles per hour. Otherwise, she drives her motorcycle at a speed of $V = 30$ miles per hour.
 - (a) Find the pmf of V and the expected value of V .
 - (b) Let the random variable T denote the time in minutes for Alice to get to class. Specify the function g such that $T = g(V)$. What is $E[T]$?
 - (c) Does $g(E[V]) = E[g(V)]$?
2. Consider a nonnegative integer-valued random variable X .
 - (a) Suppose X takes on values 1, 2, or 3 only. Show that

$$E[X] = \sum_{k=0}^2 P\{X > k\}$$

Hint: Write out $E[X]$ as a sum of 6 values of $p_X(u)$ and re-arrange the sum.

- (b) More generally, show that for nonnegative integer-valued random variables X :

$$E[X] = \sum_{k=0}^{\infty} P\{X > k\}.$$

Hint: re-arrange a sum again (but more systematically than you did in part (a)).

- (c) Let X be a *geometric random variable* with parameter p . Find $P\{X > k\}$ for $k \geq 0$ and then use the formula proved in part (b) to find $E[X]$.
 - (d) For the geometric random variable of part (c), compute $E[X]$ directly as $\sum_{k=1}^{\infty} k \cdot p_X(k)$.
 Hint: read the solution to Problem 1(g) of Problem Set 1.
3. A Megabucks lottery ticket for a weekly lottery bears one of the numbers $1, 2, \dots, 32$ on it and costs \$1. Assume that all 32 numbers are equally likely to be the winning number, and the winning number pays \$29, i.e., \$28 winnings plus your entry fee back.
 - (a) What is the average amount you win (lose) on each Megabucks lottery ticket?
 - (b) You buy a ticket bearing your lucky number 7 each week until either you have won the lottery at which point you stop buying tickets, or you have purchased tickets for 29 draws without a win at which point you quit in disgust. Compare the average change in your wealth to $29 \times$ (your answer to part (a)).
 - (c) A Minibucks lottery ticket also bears one of the numbers $1, 2, \dots, 32$ on it and costs \$1. If the number on your ticket is the winning number on one of the next 21 draws of the weekly Megabucks lottery, you cash in your ticket for \$2, i.e., \$1 winnings plus your entry fee back. What is the average amount you win (lose) on a Minibucks lottery ticket?

4. Consider a quiz game in which you are given two Questions. You think you can answer Question 1 correctly with probability 0.8 and Question 2 correctly with probability 0.5. You win \$100 for answering Question 1 correctly and \$200 for answering Question 2 correctly. (An incorrect answer means you win nothing.) You can choose which Question you want to answer. Let X denote the amount that you win.
- (a) Calculate the pmf and expectation of X if you choose Question 1.
 - (b) Calculate the pmf and expectation of X if you choose Question 2.
 - (c) Which Question should you choose to answer to maximize $E[X]$?

In a variation of the quiz game, you answer your chosen Question, *and if you answer correctly, then and only then do you get to answer the other Question* and possibly win the prize for answering the other Question correctly. If your chosen Question is answered incorrectly, the quiz game terminates and you go back to your seat with winnings of \$0. Let Y denote your winnings in this game

- (d) Calculate the pmf and expectation of Y if you choose Question 1 to answer first.
- (e) Calculate the pmf and expectation of Y if you choose Question 2 to answer first.
- (f) Which Question should you choose to answer first to maximize $E[Y]$?
- (g) Write a brief explanation (understandable by someone who knows nothing of pmfs) as to why your choice of which Question to answer (first) is different in the two games described above.
- (h) More generally, let p_1 and p_2 denote the probabilities of correctly answering Questions 1 and 2 respectively, and v_1 and v_2 the prizes for correctly answering Questions 1 and 2 respectively. Show that the optimal strategy for the second game (with winnings Y) is to answer Question 1 first if and only if

$$\frac{p_1 v_1}{1 - p_1} \geq \frac{p_2 v_2}{1 - p_2}.$$

5. In the game of Chuck-A-Luck played at fairs and carnivals in the MidWest, bets are placed on numbers 1, 2, 3, 4, 5, 6, and then three fair dice are rolled. If the number chosen does not show up on any of the three dice, the bettor loses his stake. Otherwise, the dealer pays the bettor one or two or three times the amount staked according as the number chosen shows up on one or two or all three of the dice. Of course, the amount of the bet is also returned to the bettor but is *not* counted as part of the *winnings* from this game. Let X denote the winnings in this game for a \$6 bet, and remember that negative values of X correspond to losses.
- (a) What are the values taken on by X ?
 - (b) What is the pmf of X ?
 - (c) What is the value of $E[X]$?
 - (d) A player splits his \$6 bet and wagers \$1 on each of the six numbers. Let Y denote the winnings of this player. Repeat parts (a)-(c) for Y . Does the splitting strategy improve the average winnings in this game?
6. A Poisson random variable X with parameter λ has pmf $p_X(k) = \frac{\lambda^k}{k!} \exp(-\lambda)$ where $k \geq 0$ is an integer and $\lambda \geq 0$.
- (a) Show that the probability mass function sums to one.
 - (b) Find $E[X]$ in terms of λ .
 - (c) Find $E[X(X - 1)]$ in terms of λ and use this result to determine $E[X^2]$ in terms of λ .
 - (d) Using results from (b) and (c), find $\text{var}(X)$ in terms of λ .
 - (e) Find $E[z^X]$ as a function of z and λ .