University of Illinois Spring 2008

ECE 313: Problem Set 3
Axioms of Probability

This Problem Set contains six problems

Due: Wednesday February 6 at the beginning of class.
Reading: Ross, Chapter 2, Sections 1-5, and Chapter 4
Noncredit Exercises: DO NOT turn these in.
Chapter 1: Problems 1-5, 7, 9;
Theoretical Exercises 4, 6, 13; Self-Test Problems 1-15.
Chapter 2: Problems 3, 4, 9, 10,11,12,14;
Theoretical Exercises 1-3, 6, 10, 11, 16, 19, 20; Self-Test Problems 1-8

1. For $\alpha > 0$, the *Gamma function* $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1}\exp(-t)\,dt$.

   (a) Show by integration that $\Gamma(1) = 1$.

   (b) Use integration by parts to show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, and use this to deduce that $\Gamma(n+1) = n!$.

   (c) If α is not an integer, show that $\Gamma(\alpha+1) = \alpha(\alpha-1)(\alpha-2)\cdots\Gamma(\alpha-\lfloor\alpha\rfloor)$ where $\lfloor\alpha\rfloor$ is the *integer part* of α. Note that $0 < \alpha - \lfloor\alpha\rfloor < 1$. For $0 < \beta < 1$, $\Gamma(\beta)$ must be evaluated by numerical integration, except...

   (d) show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hint: make the change of variables $\sqrt{2t} = x$. Then show that
   $$\Gamma\left(\frac{1}{2}\right) = \left(2\int_0^\infty\int_0^\infty \exp(-[x^2+y^2]/2)\,dx\,dy\right)^{1/2}$$
   where the double integral can be evaluated by a change to polar coordinates as in Problem 4(b) of Problem Set 1.

2. The binomial coefficient $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the number of subsets of size k drawn from a set of size n.

   (a) Give a combinatorial proof that $\displaystyle\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$. Hint: How many committees of size k can be formed from a group of m men and n women?

   (b) What is the coefficient of x^k in the polynomial $(1+x)^{m+n}$?
   Now write $(1+x)^{m+n} = (1+x)^m(1+x)^n$ and find the coefficient of x^k on the right hand side in terms of the coefficients of $(1+x)^m$ and $(1+x)^n$. If you have taken ECE 410, you might recognize a discrete convolution here ....

   (c) Use the result of part (a) to show that $\displaystyle\binom{2n}{n} = \sum_{i=0}^{n}\left[\binom{n}{i}\right]^2$.

3. Ross, Chapter 2, Problem 13 (page 57).

4. Find $P(A \cup (B^c \cup C^c)^c)$ in each of the following four cases:
   (a) A, B, and C are mutually exclusive events and $P(A) = 1/3$.
   (b) $P(A) = 2P(B \cap C) = 4P(A \cap B \cap C) = 1/2$.
   (c) $P(A) = 1/2,\ P(B \cap C) = 1/3$, and $P(A \cap C) = 0$.
   (d) $P(A^c \cap (B^c \cup C^c)) = 0.6$.

5. ["Eat your broccoli, dear; it's good for you"]
   (a) Your mother has bought three servings of broccoli and two servings of cauliflower for next week (Monday through Friday) and will serve one vegetable each day.
      i. Define an appropriate sample space that includes all possible outcomes of this experiment (which you don't get to perform: it is performed on you!). Assume that all the outcomes are equally likely.
      ii. What is the probability of having broccoli on Monday?
      iii. What is the probability of having broccoli on Monday and Friday?
      iv. What is the probability of having broccoli on Monday, Wednesday, and Friday?
   (b) Suppose instead that A, B, and C denote the events that your mother serves asparagus, broccoli or cauliflower for dinner. From (bitter?) experience, you know that these events are mutually exclusive and that $P(A) = 0.2$, $P(B) = 0.5$, and $P(C) = 0.3$. Each day is an *independent trial*, that is, your mother, a lady of formidable temperament albeit limited culinary skills, makes *independent decisions* as to which vegetable to serve without taking into account your opinion that Cheetos is a vegetable that goes well with any entree. Over a three day period, what is the probability that
      i. she serves the same vegetable on all three days?
      ii. she serves the same vegetable exactly two days out of three?
      iii. she serves different vegetables on the three days?
   (c) **Optional for 0 Comp II credit:** Write a 500-word essay on why you like dorm food so much.

6. (a) $\Omega = \{0, 1, 2, \ldots\}$ is a countably infinite sample space with $P(n) = \dfrac{(\ln 2)^n}{2(n!)}$ for all $n \geq 0$. Remember that $0! = 1$.
      i. Show that $P(\Omega) = 1$ for this probability assignment.
      ii. Prove that the probability that the outcome is an even number is $5/8$. Remember that 0 is an even number.
   (b) Bob & Carol & Ted & Alice take turns (in that order) tossing a coin with $P(H) = p$, $0 < p < 1$. The first one to toss a Head wins the game. Calculate their *win probabilities* $P(B), P(C), P(T)$, and $P(A)$ and show that
      i. $P(B) > P(C) > P(T) > P(A)$.
      ii. $P(B) + P(C) + P(T) + P(A) = 1$.