University of Illinois Spring 2008

ECE 313: Problem Set 2

Due: Wednesday January 30 at the beginning of class.
Reading: Ross, Chapters 1 and 2

This Problem Set contains seven problems

1. Consider events O and G pertaining to the outcome of the presidential party primaries, where event O corresponds to Barack Obama winning the South Carolina democratic primary and event G corresponds to Rudy Giuliani winning the Florida republican primary. If the probability that at least one of the two events occurs is 0.65 and the probability that at least one of the events does not occur is 0.7, what is the probability that exactly one of the two events will occur?

2. Write down the first five terms in the expansions of $(1+x)^n$ and $(1-x)^n$ (preferably on two consecutive lines and with coefficients of x^k on the two lines being one above the other in vertical alignment.) Now show that exactly 2^{n-1} of the 2^n subsets of a sample space of size n contain an even number of elements (the other 2^{n-1} subsets contain an odd number of elements). Hint: remember that 0 is an even number and set $x = 1$ in $(1+x)^n \pm (1-x)^n$.

3. Prove each of the inequalities below, and find the conditions under which equality holds in each of the inequalities:

   (a) For events A and B:
   $$P(A \cap B) \geq P(A) + P(B) - 1.$$

   (b) For events A and B:
   $$\frac{P(A) + P(B)}{2} \leq P(A \cup B) \leq P(A) + P(B)$$
   and for events A, B, and C:
   $$\frac{P(A) + P(B) + P(C)}{3} \leq P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$$

4. The University of Illinois "Fans of Michigan Despite Living in Urbana" (FOMDLIU) club consists of a leader and a number (possibly zero) of additional club members.

   (a) Suppose we have identified $n > 0$ fearless individuals who might potentially form FOMDLIU. Explain why the number of possible FOMDLIU clubs is $n2^{n-1}$.

   (b) For $1 \leq k \leq n$, how many FOMDLIU clubs have exactly k members? Find a different way of counting the number of possible FOMDLIU clubs and establish that
   $$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$

(c) Now differentiate $(1+x)^n$ via the chain rule, and also by expanding out $(1+x)^n$ via the binomial theorem and then differentiating term by term. Set $x = 1$ to prove the identity of part (b) again.

5. The manufacturer of a cereal tests samples from the production line to see if the samples snap, crackle, and pop as advertised. Let A, B, and C respectively denote the events that the sample being tested *does not* snap, *does not* crackle, and *does not* pop. The manufacturer's tests show that $P(A) = P(C) = 0.3$, $P(B) = 0.2$, $P(AB \cup AC \cup BC) = 0.3$, $P(A \cap B \cap C) = 0.05$, $P(A \cap B) = 0.1$, and $P(A \cap C) = 2P(B \cap C)$.

   (a) Sketch the sample space and indicate on it the events A, B, and C.

   (b) What is the probability that the cereal snaps, crackles, and pops?

   (c) Cereal that fails exactly one test is sold to discount supermarket chains to be marketed under the names Soggies, Blecchies, and Mushies. What is the probability that the sample fails *only* the snap test? *only* the crackle test? *only* the pop test?

6. Express each of the following events in terms of the events A, B, and C, and the operations of complementation, union, and intersection:

   (a) at least one of the events A, B, C occurs;

   (b) at most one of the events A, B, C occurs;

   (c) none of the events A, B, C occurs;

   (d) all three events A, B, C occur;

   (e) exactly one of the events A, B, C occurs;

   (f) events A and B occur, but not C;

   (g) either event A occurs, or if not then B also does not occur.

   In each case draw the corresponding Venn diagrams or Karnaugh maps, whichever is more convenient.

7. Five basketball teams (call them A, B, C, D, and E) play a round-robin tournament in which each team plays each of the other four exactly once.

   (a) How many basketball games are there in this tournament? More generally, how many games would there be in an n-team round-robin tournament?

   (b) Is it possible to arrange matters so that each of the five teams wears dark-colored (away) uniforms for two games and light-colored (home) uniforms for two games? If so, exhibit a schedule of games specifying the colors worn by each team.
   Basketball nonfanatics are reminded that in every basketball game, one team wears dark-colored uniforms and the other wears light-colored uniforms.

   (c) Now suppose that each game is equally likely to end in a win for either team.

      i. What is the probability that some team wins all four of its games?

      ii. What is the probability that some team loses all four of its games?

      iii. What is the probability that one team wins all four of its games *and* another loses all four of its games?

      iv. What is the probability that one team wins all four of its games *and* another loses all four of its games *and* the remaining teams have identical 2-2 records?