ECE 313: Solutions to Problem Set 13

1. (a) $f_{\mathcal{X}^2}(v) = \dfrac{1}{2\sqrt{v}}\left[f_{\mathcal{X}}(\sqrt{v}) + f_{\mathcal{X}}(-\sqrt{v})\right] = \dfrac{1}{2\sqrt{v}} \times \dfrac{1}{\sigma\sqrt{2\pi}}\left[\exp(-v/2\sigma^2) + \exp(-v/2\sigma^2)\right]$
$= \dfrac{1}{\sigma\sqrt{2v}} \times \dfrac{1}{\sqrt{\pi}} \exp\left(-\dfrac{v}{2\sigma^2}\right) = \dfrac{\lambda(\lambda v)^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)} \exp(-\lambda v)$ where $\lambda = \dfrac{1}{2\sigma^2}$ and $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.
Hence, $\mathcal{X}^2 \sim \mathsf{Gamma}\left(\dfrac{1}{2}, \dfrac{1}{2\sigma^2}\right)$.

(b) The sum of independent $\mathsf{Gamma}(t_i, \lambda)$ random variables is a $\mathsf{Gamma}(\sum t_i, \lambda)$ random variable. Hence, $\mathcal{W} = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$ is a $\mathsf{Gamma}\left(\dfrac{3}{2}, \dfrac{1}{2\sigma^2}\right)$ random variable whose pdf is $f_{\mathcal{W}}(\alpha) = \begin{cases} \frac{1}{\sigma^3}\sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{\alpha}{2\sigma^2}\right), & \alpha \geq 0, \\ 0, & \alpha < 0. \end{cases}$
If $\sigma^2 = 4$, $f_{\mathcal{W}}(5) = (1/8)\sqrt{5/2\pi}\exp(-5/8)$.

(c) $\mathsf{E}[\mathcal{W}] = \mathsf{E}[\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2] = \mathsf{E}[\mathcal{X}^2] + \mathsf{E}[\mathcal{Y}^2] + \mathsf{E}[\mathcal{Z}^2] = 3\sigma^2$ since $\mathcal{W} \sim \mathsf{Gamma}(3/2, 1/2\sigma^2))$, and its expected value is the ratio of the parameters, *viz.* $3\sigma^2$.

(d) The pdf of $\mathcal{H} = \frac{1}{2}m\mathcal{W}$ is $f_{\mathcal{H}}(\beta) = (2/m)f_{\mathcal{W}}(2\beta/m)$. Since $\sigma^2 = \frac{kT}{m}$, we get that the kinetic energy $\mathcal{H}$ has the Maxwell-Boltzmann pdf: $f_{\mathcal{H}}(\beta) = \dfrac{2}{\sqrt{\pi}}(kT)^{-\frac{3}{2}}\sqrt{\beta}\exp\left(-\dfrac{\beta}{kT}\right)$ for $\beta \geq 0$.

(e) $F_{\mathcal{V}}(\gamma) = P\{\mathcal{V} \leq \gamma\} = P\{\mathcal{W} \leq \gamma^2\} = F_{\mathcal{W}}(\gamma^2)$. Hence,

$$f_{\mathcal{V}}(\gamma) = 2\gamma f_{\mathcal{W}}(\gamma^2) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right) \text{ for } \gamma \geq 0.$$

(f) $\mathsf{E}[\mathcal{V}] = \displaystyle\int_0^\infty \gamma \cdot \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right)d\gamma = \int_0^\infty 2\sqrt{\frac{2kT}{m\pi}}x\cdot\exp(-x)\,dx = 2\sqrt{\frac{2kT}{m\pi}}$
on substituting $m\gamma^2/2kT = x$. Alternatively,
$\mathsf{E}[\mathcal{V}] = \mathsf{E}[\sqrt{\mathcal{W}}] = \displaystyle\int_0^\infty \sqrt{\alpha}\frac{1}{\sigma^3}\sqrt{\frac{\alpha}{2\pi}}\exp\left(-\frac{\alpha}{2\sigma^2}\right)d\alpha = \int_0^\infty \frac{4\sigma}{\sqrt{2\pi}}x\cdot\exp(-x)\,dx = \frac{4\sigma}{\sqrt{2\pi}} =$
$2\sqrt{\dfrac{2kT}{m\pi}}$ on substituting $\alpha/2\sigma^2 = x$ and remembering that $\sigma^2 = \frac{kT}{m}$.

2. $\mathsf{E}[\mathcal{X}] = 1, \mathsf{E}[\mathcal{Y}] = 4, \mathsf{var}(\mathcal{X}) = 4, \mathsf{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X},\mathcal{Y}} = 0.1$.

(a) $\mathsf{E}[\mathcal{Z}] = \mathsf{E}[2(\mathcal{X}+\mathcal{Y})(\mathcal{X}-\mathcal{Y})] = 2\mathsf{E}[\mathcal{X}^2 - \mathcal{Y}^2] = 2\mathsf{E}[\mathcal{X}^2] - 2\mathsf{E}[\mathcal{Y}^2] = 2[4+1^2] - 2[9+4^2] = -40$.

(b) $\mathsf{cov}(\mathcal{T},\mathcal{U}) = \mathsf{cov}(2\mathcal{X}+\mathcal{Y}, 2\mathcal{X}-\mathcal{Y}) = 4\cdot\mathsf{cov}(\mathcal{X},\mathcal{X}) + 2\cdot\mathsf{cov}(\mathcal{Y},\mathcal{X}) - 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y}) - \mathsf{cov}(\mathcal{Y},\mathcal{Y})$
$= 4\cdot\mathsf{var}(\mathcal{X}) + 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y}) - 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y}) - \mathsf{var}(\mathcal{Y}) = 4\cdot\mathsf{var}(\mathcal{X}) - \mathsf{var}(\mathcal{Y}) = 4\cdot 4 - 9 = 7$.

(c) $\mathsf{E}[\mathcal{W}] = \mathsf{E}[3\mathcal{X} + \mathcal{Y} + 2] = 3\mathsf{E}[\mathcal{X}] + \mathsf{E}[\mathcal{Y}] + 2 = 9$.
$\mathsf{var}(\mathcal{W}) = \mathsf{var}(3\mathcal{X}+\mathcal{Y}+2) = 3^2\cdot\mathsf{var}(\mathcal{X}) + \mathsf{var}(\mathcal{Y}) + 2\cdot 3\cdot 1\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y}) = 9\cdot 4 + 9 + 6\cdot 2\cdot 3\cdot 0.1 = 48.6$.

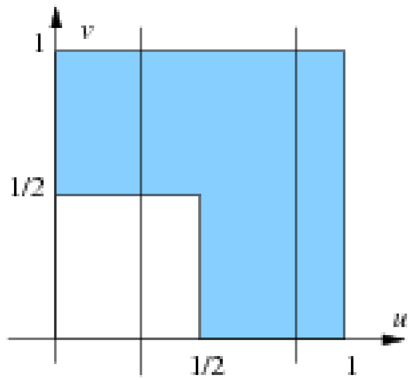
(d) $P\{\mathcal{W} > 0\} = 1 - \Phi\left(\dfrac{0-9}{\sqrt{48.6}}\right) = 1 - \Phi\left(-\dfrac{9}{\sqrt{48.6}}\right) = \Phi\left(\dfrac{9}{\sqrt{48.6}}\right)$.

3. (a) $\mathsf{var}(\mathcal{X}+\mathcal{Y}) = \mathsf{var}(\mathcal{X}) + \mathsf{var}(\mathcal{Y}) + 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y}) = 36$.
$\mathsf{var}(\mathcal{X}-\mathcal{Y}) = \mathsf{var}(\mathcal{X}) + \mathsf{var}(\mathcal{Y}) - 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y}) = 64$. Hence, $\mathsf{cov}(\mathcal{X},\mathcal{Y}) = -7$.
From the above, $2\cdot\mathsf{var}(\mathcal{X}) + 2\cdot\mathsf{var}(\mathcal{Y}) = 8\cdot\mathsf{var}(\mathcal{Y}) = 100$, giving $\mathsf{var}(\mathcal{Y}) = 12.5, \mathsf{var}(\mathcal{X}) = 37.5$ and $\rho_{\mathcal{X},\mathcal{Y}} = \mathsf{cov}(\mathcal{X},\mathcal{Y})/\sqrt{\mathsf{var}(\mathcal{X})\mathsf{var}(\mathcal{Y})} = -7/12.5\sqrt{3}$.

(b) $\mathsf{var}(\mathcal{X}+\mathcal{Y}) = \mathsf{var}(\mathcal{X}) + \mathsf{var}(\mathcal{Y}) + 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y})$ equals $\mathsf{var}(\mathcal{X}-\mathcal{Y}) = \mathsf{var}(\mathcal{X}) + \mathsf{var}(\mathcal{Y}) - 2\cdot\mathsf{cov}(\mathcal{X},\mathcal{Y})$ if and only if $\mathsf{cov}(\mathcal{X},\mathcal{Y}) = 0$, that is, if and only if $\mathcal{X}$ and $\mathcal{Y}$ are uncorrelated.

(c) No, whether $\mathsf{var}(\mathcal{X})$ equals $\mathsf{var}(\mathcal{Y})$ or not has no bearing on the question of whether $\mathsf{cov}(\mathcal{X},\mathcal{Y})$ is zero or not.

4. The random point $(\mathcal{X},\mathcal{Y})$ is uniformly distributed on the shaded region shown in the figure below. Clearly, $f_{\mathcal{X},\mathcal{Y}}(u,v) = \dfrac{4}{3}$ on this region.



(a) $f_{\mathcal{X}}(u)$ is the *area of the cross-section of the pdf surface* along the line u. There are two cases to be considered, as shown in the left-hand figure above. It is obvious almost by inspection that $f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \le u \le \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

$$\mathsf{E}[\mathcal{X}] = \int_{-\infty}^{\infty} u\cdot f_{\mathcal{X}}(u)\,du = \int_0^{1/2} u\cdot\frac{2}{3}\,du + \int_{1/2}^{1} u\cdot\frac{4}{3}\,du = \frac{7}{12}$$

$$\mathsf{E}[\mathcal{X}^2] = \int_{-\infty}^{\infty} u^2\cdot f_{\mathcal{X}}(u)\,du = \int_0^{1/2} u^2\cdot\frac{2}{3}\,du + \int_{1/2}^{1} u^2\cdot\frac{4}{3}\,du = \frac{5}{12}$$

$$\mathsf{var}(\mathcal{X}) = \mathsf{E}[\mathcal{X}^2] - (\mathsf{E}[\mathcal{X}])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

(b) By symmetry, $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$ are the same function: $f_{\mathcal{Y}}(v) = \begin{cases} \frac{2}{3}, & 0 \le v \le \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < v \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

$\mathsf{E}[\mathcal{Y}] = \dfrac{7}{12}$, $\mathsf{var}(\mathcal{Y}) = \dfrac{11}{144}$

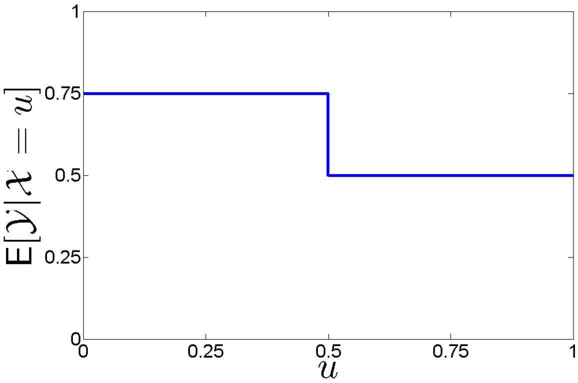
(c) Given that $\mathcal{X} = \alpha$, the conditional pdf $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha)$ is the cross-section of the pdf surface at $u = \alpha$ unitized to have area 1.
For $0 < \alpha < \frac{1}{2}$: $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) \sim \mathsf{Uniform}[\frac{1}{2}, 1] \Rightarrow \mathsf{E}[\mathcal{Y}|\mathcal{X}=\alpha] = \frac{3}{4}$, $\mathsf{var}[\mathcal{Y}|\mathcal{X}=\alpha] = \frac{1}{48}$
For $\frac{1}{2} < \alpha < 1$: $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) \sim \mathsf{Uniform}[0, 1] \Rightarrow \mathsf{E}[\mathcal{Y}|\mathcal{X}=\alpha] = \frac{1}{2}$, $\mathsf{var}[\mathcal{Y}|\mathcal{X}=\alpha] = \frac{1}{12}$

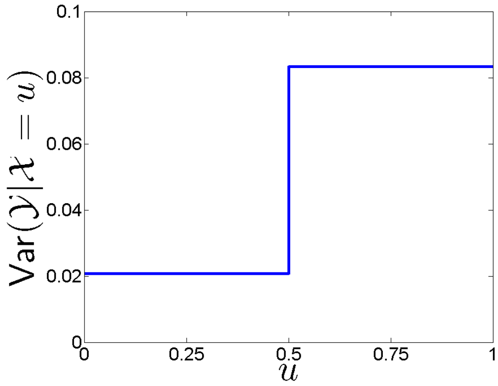
(d) $0 \le v \le \dfrac{1}{2} : f_{\mathcal{Y}}(v) = \displaystyle\int_0^1 f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) f_{\mathcal{X}}(\alpha)\,d\alpha = \int_0^{1/2} (0)\cdot\frac{2}{3}\,d\alpha + \int_{1/2}^{1} (1)\cdot\frac{4}{3}\,d\alpha = \frac{2}{3}$

$\dfrac{1}{2} \le v \le 1 : f_{\mathcal{Y}}(v) = \displaystyle\int_0^1 f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) f_{\mathcal{X}}(\alpha)\,d\alpha = \int_0^{1/2} (2)\cdot\frac{2}{3}\,d\alpha + \int_{1/2}^{1} (1)\cdot\frac{4}{3}\,d\alpha = \frac{4}{3}$

Yes, we get the same answer as in part (b).

(e) See figures below.





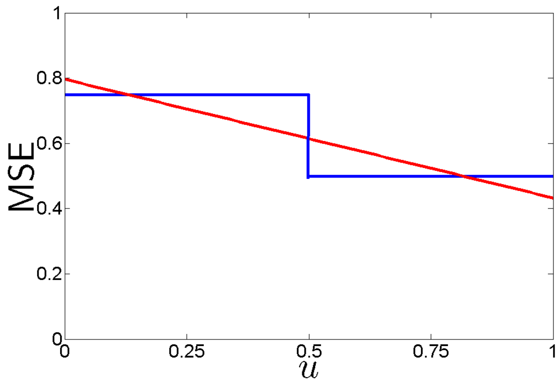
(f) MSE of MMSE estimator $= \mathsf{E}\left[\mathsf{var}(\mathcal{Y}|\mathcal{X}=u)\right] = \int_0^{1/2} \frac{1}{48}\cdot\frac{2}{3}\,du + \int_{1/2}^{1} \frac{1}{12}\cdot\frac{4}{3}\,du = \frac{7}{432}$

(g) $\mathsf{E}[\mathcal{X}\mathcal{Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv f_{\mathcal{X},\mathcal{Y}}(u,v)\,du\,dv = \frac{4}{3}\left[\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} uv\,du\,dv + \int_{0}^{1}\int_{\frac{1}{2}}^{1} uv\,du\,dv\right] = \frac{5}{16}$

$\mathsf{cov}(\mathcal{X},\mathcal{Y}) = \mathsf{E}[\mathcal{X}\mathcal{Y}] - \mathsf{E}[\mathcal{X}]\mathsf{E}[\mathcal{Y}] = \frac{5}{16} - \left(\frac{7}{12}\right)^2 = -\frac{1}{36}$

$\rho_{\mathcal{X},\mathcal{Y}} = \frac{\mathsf{cov}(\mathcal{X},\mathcal{Y})}{\sigma_{\mathcal{X}}\sigma_{\mathcal{Y}}} = \frac{-1/36}{11/144} = -\frac{4}{11}$

(h) $\widehat{\mathcal{Y}} = \mathsf{E}[\mathcal{X}] + \rho_{\mathcal{X},\mathcal{Y}}\sqrt{\mathsf{var}(\mathcal{Y})/\mathsf{var}(\mathcal{X})}\,(u - \mathsf{E}[\mathcal{X}]) = -\frac{4}{11}u + \frac{35}{44}.$



MSE of linear estimator $= \mathsf{var}(\mathcal{Y})\left(1 - \rho^2_{\mathcal{X},\mathcal{Y}}\right) = \frac{11}{144}\left(1 - \left(\frac{4}{11}\right)^2\right) = \frac{35}{528} > \frac{7}{432}$

5. This problem considers the following situation. A random point in the plane has coordinates $(\mathcal{X},\mathcal{Y})$ with respect to our chosen axes, and these coordinates are jointly Gaussian random variables. If we rotate the axes by an angle θ, the coordinates of the same random point *with respect to the new axes* are $(\mathcal{Z},\mathcal{W})$. The joint pdf surface is still the same, but the coordinate axes are different, and hence the mathematical formula expressing the value of the joint pdf is different: in fact, it is a jointly Gaussian pdf with means, variances, and covariance given by the answers to be found in parts (a) and (b).

   Note that $\mathsf{E}[\mathcal{X}] = \mathsf{E}[\mathcal{Y}] = 0$ and hence $\mathsf{E}[\mathcal{X}^2] = \sigma_1^2$ and $\mathsf{E}[\mathcal{Y}^2] = \sigma_2^2$ while $\mathsf{cov}[\mathcal{X},\mathcal{Y}] = \rho\cdot\sigma_1\sigma_2$.

   (a) $\mathsf{E}[\mathcal{Z}] = \mathsf{E}[\mathcal{X}\cos\theta + \mathcal{Y}\sin\theta] = \mathsf{E}[\mathcal{X}]\cos\theta + \mathsf{E}[\mathcal{Y}]\sin\theta = 0.$
   $\mathsf{E}[\mathcal{W}] = \mathsf{E}[\mathcal{Y}\cos\theta - \mathcal{X}\sin\theta] = 0$ also.

Since $\mathcal{Z}$ and $\mathcal{W}$ are zero-mean random variables, we get
$\mathsf{var}(\mathcal{Z}) = \mathsf{E}[\mathcal{Z}^2] = \mathsf{E}[\mathcal{X}^2\cos^2\theta + \mathcal{Y}^2\sin^2\theta + 2\mathcal{X}\mathcal{Y}\sin\theta\cos\theta] = \sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta + 2\rho\sigma_1\sigma_2\sin\theta\cos\theta$, and
$\mathsf{var}(\mathcal{W}) = \mathsf{E}[\mathcal{W}^2] = \mathsf{E}[\mathcal{X}^2\sin^2\theta + \mathcal{Y}^2\cos^2\theta - 2\mathcal{X}\mathcal{Y}\sin\theta\cos\theta] = \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta - 2\rho\sigma_1\sigma_2\sin\theta\cos\theta$.

(b) Since $\mathcal{Z}$ and $\mathcal{W}$ are zero-mean random variables,

$$\begin{aligned}
\mathsf{cov}(\mathcal{Z},\mathcal{W}) = \mathsf{E}[\mathcal{Z}\mathcal{W}] &= \mathsf{E}[(\mathcal{X}\cos\theta + \mathcal{Y}\sin\theta)(\mathcal{Y}\cos\theta - \mathcal{X}\sin\theta)] \\
&= \mathsf{E}[\mathcal{Y}^2]\sin\theta\cos\theta - \mathsf{E}[\mathcal{X}^2]\sin\theta\cos\theta + \mathsf{E}[\mathcal{X}\mathcal{Y}](\cos^2\theta - \sin^2\theta) \\
&= \sin\theta\cos\theta\cdot(\sigma_2^2 - \sigma_1^2) + (\cos^2\theta - \sin^2\theta)\cdot\rho\sigma_1\sigma_2 \\
&= \frac{1}{2}\cdot\sin 2\theta\cdot(\sigma_2^2 - \sigma_1^2) + \cos 2\theta\cdot\rho\sigma_1\sigma_2
\end{aligned}$$

(c) $\mathcal{Z}$ and $\mathcal{W}$ are jointly Gaussian random variables and thus they are independent if $\mathsf{cov}(\mathcal{Z},\mathcal{W}) = 0$. We get independent random variables if we choose $\theta = \dfrac{1}{2}\arctan\left(\dfrac{2\rho\cdot\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right)$
Note that if θ_0 is a solution to this equation, then $\theta_0 + \pi$ is also a solution, as are $\theta_0 \pm \pi/2$. That is, there are *four* different values of θ in the range $[0, 2\pi)$ that can be used to get independent Gaussian random variables from $\mathcal{X}$ and $\mathcal{Y}$. In particular, if $\sigma_1 = \sigma_2$, then θ can take on values $\pm\pi/4$ and $\pm 3\pi/4$.