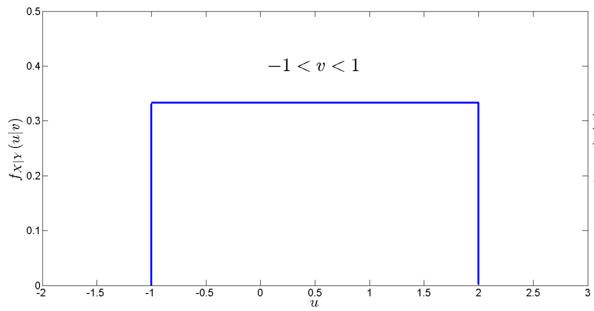
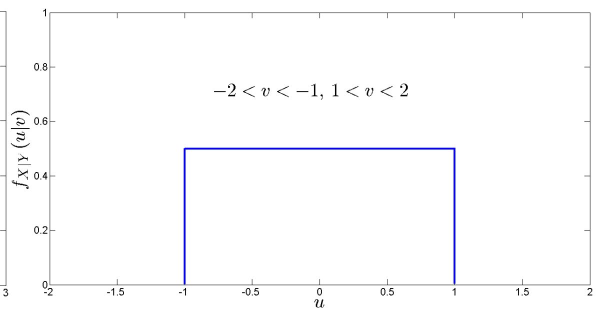
University of Illinois Spring 2008

ECE 313: Solutions to Problem Set 12

1. (a) For $1 < |v| < 2$, the cross-section of the joint pdf surface along the line v is a rectangle extending from $u = -1$ to $u = 1$. Therefore the conditional pdf of X given Y is Uniform$[-1, 1]$.
Similarly for $|v| < 1$, the cross-section extends from $u = -1$ to $u = 2$, so the conditional pdf of X given Y is Uniform$[-1, 2]$.



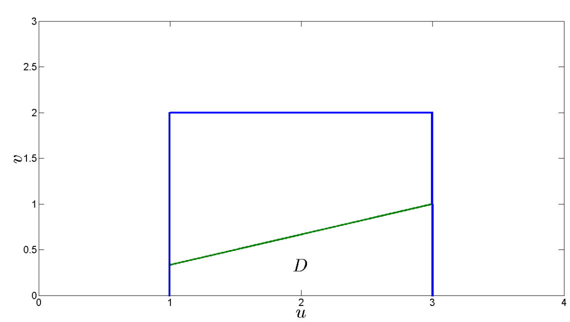


(b) Using the uniformity of the conditional pdfs, we have:
$$\mathsf{E}[X|Y = v] = \begin{cases} 0, & 1 \le |v| \le 2 \\ \frac{1}{2}, & |v| \le 1 \end{cases} = \text{rect}(v/2), \quad \mathsf{Var}[X|Y = y] = \begin{cases} 1/3, & 1 \le |v| \le 2 \\ 3/4, & |v| \le 1 \end{cases}$$

(c) It is easy to see that $f_Y(v) = \begin{cases} 0.2, & 1 < |v| < 2, \\ 0.3, & |v| < 1, \\ 0, & \text{otherwise.} \end{cases}$ Hence,
$$\mathsf{E}[X] = \int_{-\infty}^{\infty} \mathsf{E}[X|Y = v] \cdot f_Y(v)\, dv \int_{-\infty}^{\infty} \text{rect}(v/2) \cdot f_Y(v)\, dv = \int_{-1}^{1} \left(\frac{1}{2}\right) \cdot (0.3)\, dv = \frac{3}{10}$$

2. (a) First, note that the support of $f_{X,Y}(u, v)$ is a square and the joint density does not depend on v. Thus, conditioned on any value of u, we see that $f_{Y|X}(v|u)$ must be uniform on $[0, 2]$.
$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u, v)\, du = \int_1^2 \left(\frac{u}{8}\right)\, du = \frac{1}{2},\ 0 \le v \le 2 \Rightarrow f_Y(v) \sim$ Uniform$[0, 2]$
Since $f_{Y|X}(v|u) = f_Y(v)$, X and Y are independent.

However, conditioned on D, X and Y are not independent. Conditioned on D, the support of $f_{X,Y|D}(u, v|D)$ is a trapezoid. We get this by intersecting the previous support (the rectangle) with the event $\left\{v \le \frac{u}{3}\right\}$ (see the diagram below). So, given D, the support of Y is $[0, 1]$. But given D and $\{X = u\}$, the support of Y is $\left[0, \frac{u}{3}\right]$, which depends on u. Thus, given D, the conditional density of Y given X and the marginal density of Y differ.

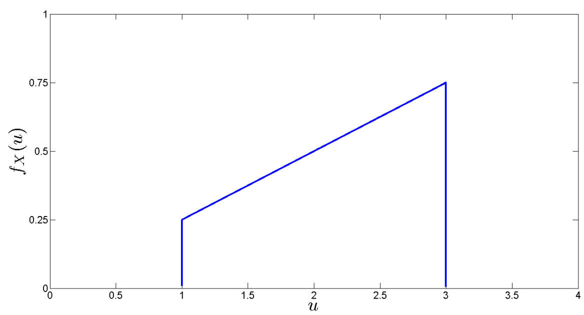


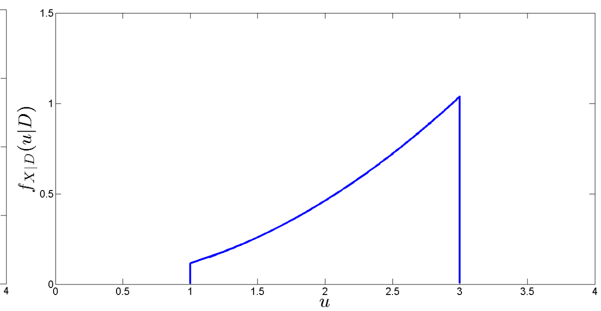
(b) $P(D) = P(R \le 1/3) = P\left(\frac{Y}{X} \le \frac{1}{3}\right) = P\left(Y \le \frac{X}{3}\right) = \int_1^3 \int_{v=0}^{u/3} \left(\frac{u}{8}\right) du = \frac{13}{36}$

(c) For $1 \le u \le 3$: $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv = \int_{v=0}^{2} \left(\frac{u}{8}\right) dv = \frac{u}{4}$

For $1 \le u \le 3$: $\int_{(u,v)\in D} f_{X,Y}(u,v)\, dv = \int_0^{u/3} \left(\frac{u}{8}\right) dv = \frac{u^2}{24}$

$$f_{X|D}(u|D) = \frac{\int_{(u,v)\in D} f_{X,Y}(u,v)\, dv}{P(D)} = \frac{u^2/24}{13/36} = \begin{cases} \frac{3u^2}{26}, & 1 \le u \le 3 \\ 0, & \text{otherwise} \end{cases}$$





(d) $F_{R|D}(r|D) = P\{R \le r|D\} = \frac{P\{R \le r, D\}}{P\{D\}}$

$$0 \le r \le \frac{1}{3} \quad : \quad P\{R \le r \cap D\} = P\{R \le r\} = P\{Y \le rX\} = \int_1^3 \int_0^{ru} \left(\frac{u}{8}\right) dv\, du = \frac{13r}{12}$$

$$r \ge \frac{1}{3} \quad : \quad P\{R \le r \cap D\} = P\left\{R \le \frac{1}{3}\right\} = P\{D\} = \frac{13}{36}$$

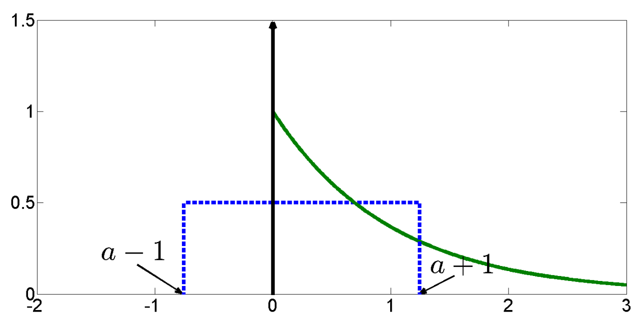
$$F_{R|D}(r|D) = \begin{cases} 0, & r \le 0 \\ 3r, & 0 \le r \le \frac{1}{3} \\ 1, & r \ge \frac{1}{3} \end{cases} \quad \Rightarrow \quad f_{R|D}(r|D) = \begin{cases} 3, & 0 \le r \le \frac{1}{3} \\ 0, & \text{otherwise} \end{cases}$$

Thus, succinctly, conditioned on D, R is uniformly distributed on $[0, 3]$.

3. The densities of X_1 and X_2 are given by:

$$f_{X_1}(u) = \begin{cases} \frac{1}{2} & -1 \le u \le 1 \\ 0 & \text{elsewhere} \end{cases}, \quad f_{X_2}(u) = \begin{cases} e^{-u} & u \ge 0 \\ 0 & u < 0 \end{cases}$$

pdf of $W = X_1 + X_2$: Since X_1 and X_2 are independent, the pdf of W is just the convolution of the individual pdfs (here, we are keeping the exponential density fixed and flipping and shifting the uniform density – see diagram below).

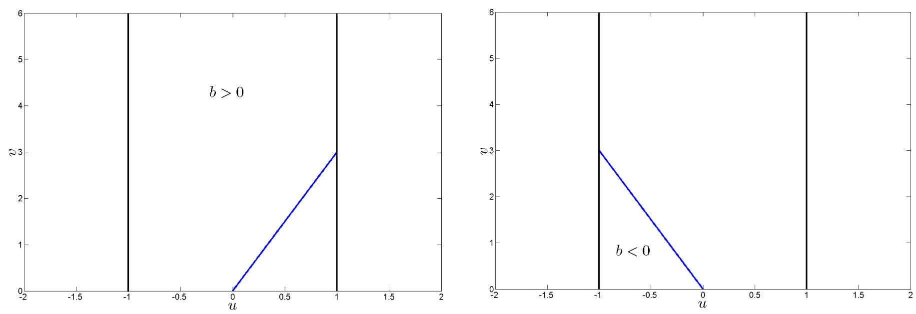


$$f_W(a) = \begin{cases} 0, & a < -1 \\ \int_0^{a+1} \frac{1}{2} e^{-u}\, du, & -1 \le a < 1 \\ \int_{a-1}^{a+1} \frac{1}{2} e^{-u}\, du, & a > 1 \end{cases} = \begin{cases} 0, & a < -1 \\ \frac{1}{2}\left(1 - e^{-(a+1)}\right), & -1 \le a < 1 \\ \frac{1}{2}(e - e^{-1})e^{-a}, & a > 1 \end{cases}$$

<u>pdf of $Z = X_1/X_2$</u>: Since $X_2 \ge 0$,

$$\begin{aligned} F_Z(b) = P\{X_1/X_2 \le b\} = P\{X_1 \le bX_2\} &= \begin{cases} \int_{u=-1}^{0} \int_{v=0}^{u/b} \frac{1}{2} e^{-v}\, dv\, du, & b < 0 \\ 1 - \int_{u=0}^{1} \int_{v=0}^{u/b} \frac{1}{2} e^{-v}\, dv\, du, & b \ge 0 \end{cases} \\ &= \begin{cases} \frac{1}{2}\left(1 + b(1 - e^{1/b})\right), & b < 0 \\ \frac{1}{2}\left(1 + b(1 - e^{-1/b})\right), & b \ge 0 \end{cases} \end{aligned}$$

The areas over which to integrate are indicated in the figures below.

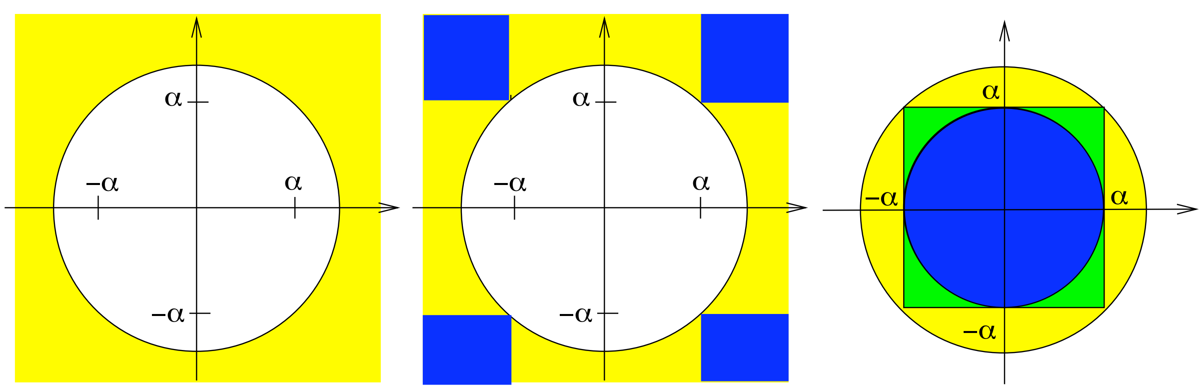


From this, the pdf of Z can be obtained by differentiating

$$f_Z(b) = \begin{cases} \frac{1}{2}\left(1 - e^{1/b} + \frac{e^{1/b}}{b}\right), & b < 0 \\ \frac{1}{2}\left(1 - e^{-1/b} - \frac{e^{-1/b}}{b}\right), & b \ge 0 \end{cases}$$

4. (a) $f_{\mathcal{X},\mathcal{Y}}(u,v) = f_{\mathcal{X}}(u) f_{\mathcal{Y}}(v) = \dfrac{1}{2\pi} \exp\left[-\dfrac{u^2+v^2}{2}\right]$.

   (b) The region over which the joint pdf must be integrated in order to find $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$ is shown in the left-hand figure below.



The volume outside the circle of radius $\sqrt{2}\alpha$ can be found by changing to polar coordinates. We have $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \displaystyle\int_{r=\sqrt{2}\alpha}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r\, d\theta\, dr = \int_{r=\sqrt{2}\alpha}^{\infty} r \exp\left(-\frac{r^2}{2}\right) dr$

$= -\exp\left(-\dfrac{r^2}{2}\right)\Big|_{\sqrt{2}\alpha}^{\infty} = \exp(-\alpha^2)$.

(c) $\mathcal{Z} = \mathcal{X}^2 + \mathcal{Y}^2$. From part (b), we get that $P\{\mathcal{Z} > \beta\} = 1 - F_{\mathcal{Z}}(\beta) = \exp(-\beta/2)$ for $\beta \geq 0$. Hence, $f_{\mathcal{Z}}(\beta) = \frac{1}{2}\exp(-\beta/2)$ for $\beta \geq 0$. This is an exponential density with parameter $\frac{1}{2}$.

(d) $P\{|\mathcal{X}| > \alpha\} = 2Q(\alpha)$. Hence, $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = P\{|\mathcal{X}| > \alpha\}P\{|\mathcal{Y}| > \alpha\} = 4Q^2(\alpha)$.

(e) From the middle figure above, it is obvious that

$$P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = 4Q^2(\alpha) < P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \exp(-\alpha^2) \text{ for } \alpha > 0.$$

(f) Taking square roots on both sides, we get $Q(x) < \frac{1}{2}\exp(-x^2/2)$ for $x > 0$.

(g) $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} = P\{|\mathcal{X}| < \alpha\}P\{|\mathcal{Y}| < \alpha\} = [1 - 2Q(\alpha)]^2$ is the probability that the random point lies inside the square of side 2α. As shown in the right-hand figure, this square is inscribed by the circle of radius α and circumscribed by the circle of radius $\sqrt{2}\alpha$. Hence,

$$P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq \alpha^2/2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 < \alpha^2\}.$$

These inequalities are equivalent to the result: $\exp(-\alpha^2) < 4Q(\alpha) - 4Q^2(\alpha) < \exp(-\alpha^2/2)$. Now, $4Q(\alpha) - 4Q^2(\alpha) < 4Q(\alpha)$, and therefore we get $Q(x) > \frac{1}{4}\exp(-x^2)$ for $x > 0$. Note that $Q(0) = \frac{1}{2}$ while the bound equals $\frac{1}{4}$ at $x = 0$. The upper bound can also be obtained from the above result. Note that since $Q(\alpha) \leq \frac{1}{2}$ for $\alpha > 0$, it follows that $4Q(\alpha) - 4Q^2(\alpha) = 4Q(\alpha)[1 - Q(\alpha)] > 2Q(\alpha)$ for $\alpha > 0$. Hence, $\exp(-\alpha^2/2) > 4Q(\alpha) - 4Q^2(\alpha) > 2Q(\alpha)$ etc.

5. (a) Converting to polar coordinates:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} cr\cdot r\,dr\,d\theta = \frac{2\pi c}{3} = 1 \Rightarrow c = \frac{3}{2\pi}$$

(b) Using LOTUS:

$$\mathsf{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{3}{2\pi}\sqrt{u^2+v^2}\cdot\frac{3}{2\pi}\sqrt{u^2+v^2}\,du\,dv = \frac{9}{4\pi^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3\,dr\,d\theta = \frac{9}{8\pi}$$

(c) $f(X,Y) = \frac{3}{2\pi}\sqrt{X^2+Y^2}$ has maximum value $\frac{3}{2\pi} < \frac{1}{2}$. Therefore $P\left\{f(X,Y) > \frac{1}{2}\right\} = 0$.