ECE 313: Solutions to Problem Set 11

1. (a) The marginal pmfs $p_{\mathcal{X}}(u)$ and $p_{\mathcal{Y}}(v)$ are column and row sums as shown in the table below.

$u \rightarrow$ / $v \downarrow$	0	1	3	5	Row sum
4	0	1/12	1/6	1/12	1/3
3	1/6	1/12	0	1/12	1/3
-1	1/12	1/6	1/12	0	1/3
Column sum	1/4	1/3	1/4	1/6	1

(b) The eyeball test tells us that $\mathcal{X}$ and $\mathcal{Y}$ are *dependent* random variables.

(c) $P\{\mathcal{X} \leq \mathcal{Y}\} = p_{\mathcal{X},\mathcal{Y}}(0,3) + p_{\mathcal{X},\mathcal{Y}}(0,4) + p_{\mathcal{X},\mathcal{Y}}(1,3) + p_{\mathcal{X},\mathcal{Y}}(1,4) + p_{\mathcal{X},\mathcal{Y}}(3,4) = \dfrac{1}{2}$.

$P\{\mathcal{X} + \mathcal{Y} \leq 4\} = p_{\mathcal{X}}(0) + p_{\mathcal{X},\mathcal{Y}}(1,3) + p_{\mathcal{X},\mathcal{Y}}(1,-1) + p_{\mathcal{X},\mathcal{Y}}(3,-1) + p_{\mathcal{X},\mathcal{Y}}(5,-1) = \dfrac{7}{12}$.

(d) $p_{\mathcal{X}|\mathcal{Y}}(u|3) = \dfrac{p_{\mathcal{X},\mathcal{Y}}(u,3)}{p_{\mathcal{Y}}(3)} = 1/2, 1/4, 1/4$ respectively for $u = 0, 1, 5$.

$\mathsf{E}[\mathcal{X}|\mathcal{Y} = 3] = 0 \times \dfrac{1}{2} + 1 \times \dfrac{1}{4} + 5 \times \dfrac{1}{4} = \dfrac{3}{2}$.

$\mathsf{var}(\mathcal{X}|\mathcal{Y} = 3) = \mathsf{E}[\mathcal{X}^2|\mathcal{Y} = 3] - (\mathsf{E}[\mathcal{X}|\mathcal{Y} = 3])^2 = 1 \times \dfrac{1}{4} + 5^2 \times \dfrac{1}{4} - \left(\dfrac{3}{2}\right)^2 = \dfrac{26}{4} - \dfrac{9}{4} = \dfrac{17}{4}$.

2. (a) $p_{\mathcal{X},\mathcal{Y}}(n,m) = p_{\mathcal{Y}|\mathcal{X}}(m|n) \cdot p_{\mathcal{X}}(n) = \binom{n}{m} p^m (1-p)^{n-m} \cdot \frac{e^{-\lambda T}(\lambda T)^n}{n!}$

$$\begin{aligned}
p_{\mathcal{Y}}(m) &= \sum_n p_{\mathcal{X},\mathcal{Y}}(n,m) \\
&= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} \cdot \frac{e^{-\lambda T}(\lambda T)^n}{n!} \\
&= \frac{p^m e^{-\lambda T}}{(1-p)^m} \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} \cdot \frac{(1-p)^n (\lambda T)^n}{n!} \\
&= \frac{p^m e^{-\lambda T}}{m!(1-p)^m} \sum_{n=m}^{\infty} \frac{[(1-p)\lambda T]^n}{(n-m)!} \cdot \frac{[(1-p)\lambda T]^m}{[(1-p)\lambda T]^m} \\
&= \frac{p^m e^{-\lambda T}(1-p)^m (\lambda T)^m}{m!(1-p)^m} \sum_{n=m}^{\infty} \frac{[(1-p)\lambda T]^{n-m}}{(n-m)!} \\
&= \frac{p^m e^{-\lambda T}(\lambda T)^m}{m!} \cdot e^{(1-p)\lambda T} \\
&= \frac{(\lambda p T)^m e^{-\lambda p T}}{m!}
\end{aligned}$$

(b) Since $\mathcal{Z} = \mathcal{X} - \mathcal{Y}$, it implies that given $\mathcal{X} = n$, the conditional pmf $p_{\mathcal{Z}|\mathcal{X}}(k|n)$ of $\mathcal{Z}$ is binomial with parameters $(n, 1-p)$. Following similar steps to those of part (a), we can show that $\mathcal{Z}$ is Poisson with parameter $\lambda(1-p)T$ or $(\lambda q)T$.

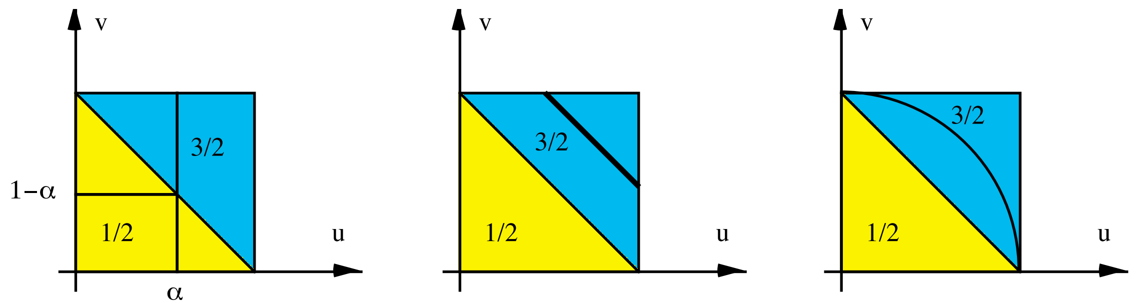
(c) When $\mathcal{Y} = m$ is observed, it must be that $\mathcal{X} \geq m$. Hence, for $n \geq m \geq 0$,

$$p_{\mathcal{X}|\mathcal{Y}}(n|m) = \frac{p_{\mathcal{Y}|\mathcal{X}}(m|n)p_{\mathcal{X}}(m)}{p_{\mathcal{Y}}(m)} = \frac{\binom{n}{m}p^m(1-p)^{n-m}\cdot(\lambda T)^n\exp(-\lambda T)/n!}{(\lambda pT)^m\cdot\exp(-\lambda pT)/m!}$$

$= \dfrac{(\lambda(1-p)T)^{n-m}\exp(-\lambda(1-p)T)}{(n-m)!}$. This is a *displaced* Poisson pmf: the conditional pmf of $\mathcal{X}$ is that of the random variable $m+\mathcal{Z}$ where $\mathcal{Z}$ is a Poisson random variable with parameter $\lambda(1-p)T$. Note that $m+\mathcal{Z}$ takes on values $m, m+1, \ldots$

(d) $\mathsf{E}[\mathcal{X}|\mathcal{Y}=m] = \mathsf{E}[m+\mathcal{Z}] = m + \lambda(1-p)T$.

3. The joint pdf is as shown in the figure below.



(a) From the left-hand figure above, $f_{\mathcal{X}}(\alpha) = 0$ for $\alpha < 0$ or $\alpha > 1$, while for any $\alpha, 0 \leq \alpha \leq 1$,

$$f_{\mathcal{X}}(\alpha) = \int_{-\infty}^{\infty} f_{\mathcal{X},\mathcal{Y}}(\alpha, v)\,dv = \int_0^{1-\alpha}\frac{1}{2}\,dv + \int_{1-\alpha}^{1}\frac{3}{2}\,dv = \frac{1}{2}(1-\alpha) + \frac{3}{2}\alpha = \frac{1}{2} + \alpha.$$

(b) When the pdf has constant value over a region, we can find the probability that the random point lies in that region by finding the area of the region and multiplying by the pdf value.

Thus, $P\{\mathcal{X}+\mathcal{Y} \leq 3/2\} = 1 - P\{\mathcal{X}+\mathcal{Y} \geq 3/2\} = 1 - \dfrac{3}{2}\times\left[\dfrac{1}{2}\times\dfrac{1}{2}\times\dfrac{1}{2}\right] = \dfrac{13}{16}$

and $P\{\mathcal{X}^2+\mathcal{Y}^2 \geq 1\} = \dfrac{3}{2}\times\left[1-\dfrac{\pi}{4}\right] = \dfrac{3}{2} - \dfrac{3\pi}{8}$.

4. The pdf is nonzero over the shaded region in the left-hand figure shown on the next page.

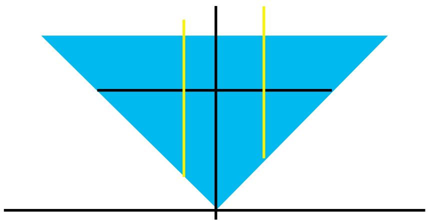
(a) , (b) From the figure, we get that for $v > 0$,

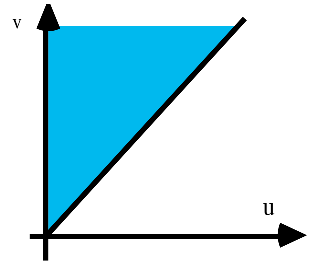
$$f_{\mathcal{Y}}(v) = \int_{u=-v}^{u=+v} c(v^2-u^2)\exp(-v)\,du = c\left[v^2u - \frac{u^3}{3}\right]\exp(-v)\bigg|_{u=-v}^{u=+v} = c\left(\frac{4}{3}\right)v^3\exp(-v)$$

which is of the form of a *gamma* pdf with parameters $(4, 1)$. Thus, $4c/3 = 1/\Gamma(4) = 1/3 \Rightarrow c = 1/8$.

On the other hand, for $u > 0$, $f_{\mathcal{X}}(u) = \displaystyle\int_{v=u}^{\infty}\frac{1}{8}(v^2-u^2)\exp(-v)\,dv = \frac{1}{4}(1+u)\exp(-u)$, while if $u < 0$, the limits are $v = -u$ and ∞. Consequently, $f_{\mathcal{X}}(u) = \frac{1}{4}(1+|u|)\exp(-u), -\infty < u < \infty$.

(c) The pdf of $\mathcal{X}$ is an even function of u and $\displaystyle\int_0^{\infty} f_{\mathcal{X}}(u)\,du$ is finite. Hence, $\mathsf{E}[\mathcal{X}] = 0$.





5. (a) The joint pdf is nonzero on the shaded region shown in the right-hand figure above.

   (b) For $u > 0$, $f_{\mathcal{X}}(u) = \displaystyle\int_v^\infty 2\exp(-u-v)\,dv = 2\exp(-2u)$ and $f_{\mathcal{X}}(u) = 0$ for $u < 0$.
   For $v > 0$, $f_{\mathcal{Y}}(v) = \displaystyle\int_0^v 2\exp(-u-v)\,du = 2\exp(-v) - 2\exp(-2v)$ and $f_{\mathcal{Y}}(v) = 0$ for $v < 0$.

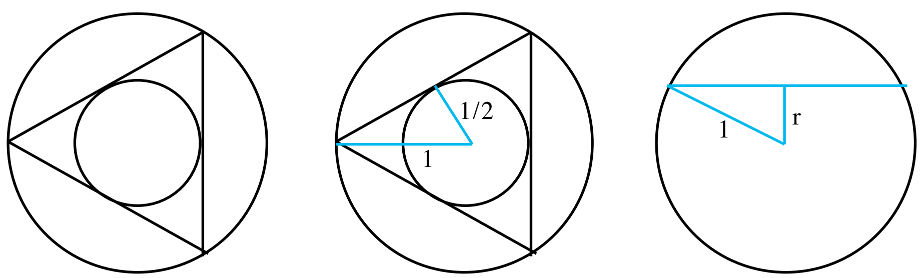
   (c) No, the eyeball test says that the random variables are dependent.

   (d) $P\{\mathcal{Y} > 3\mathcal{X}\} = \displaystyle\int_{u=0}^\infty \int_{v=3u}^\infty 2(\exp(-u-v)\,dv\,du = \int_{u=0}^\infty 2\exp(-4u)\,du = \frac{1}{2}.$

   (e) For $\alpha > 0$,
   $P\{\mathcal{X} + \mathcal{Y} \le \alpha\} = \displaystyle\int_{u=0}^{\alpha/2}\int_{v=u}^{\alpha-u} 2e^{-u-v}\,dv\,du = \int_{u=0}^{\alpha/2} 2e^{-u}[e^{(-u)} - e^{(-\alpha+u)}]\,du$
   $= 1 - \alpha\exp(-\alpha) - \exp(-\alpha).$

   (f) $f_{\mathcal{Z}}(\alpha) = \dfrac{d}{d\alpha}F_{\mathcal{Z}}(\alpha) = \dfrac{d}{d\alpha}P\{\mathcal{X}+\mathcal{Y} \le \alpha\} = \dfrac{d}{d\alpha}1 - \alpha\exp(-\alpha) - \exp(-\alpha) = \alpha\exp(-\alpha)$ for $\alpha > 0$ and $f_{\mathcal{Z}}(\alpha) = 0$ for $\alpha < 0$. This is a *gamma* pdf with parameters $(2, 1)$.

6. The pdf of $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the unit disc, and thus has value π^{-1}.



sin(π/6)=1/2

   (a) The circle inscribed in the triangle has radius 1/2 (cf. Ross, p. 217). If the midpoint of the chord is outside the circle, then the length $\mathcal{L}$ of the random chord is greater than the side of the equilateral triangle *outside* the inscribed circle in the triangle. This inscribed circle has radius 1/2 (Ross, p. 217). and hence, $P\{\mathcal{L} > \frac{1}{2}\} = \left[\pi\left(\frac{1}{2}\right)^2\right]\pi^{-1} = \frac{1}{4}$

   (b) The length of a chord at distance r from the center is $2\sqrt{1-r^2}$.
   Hence, $\mathcal{L} = 2\sqrt{1-\mathcal{R}^2} = 2\sqrt{1-\mathcal{X}^2-\mathcal{Y}^2}$
   and $F_{\mathcal{L}}(\alpha) = P\{\mathcal{L} \le \alpha\} = P\{\mathcal{X}^2 + \mathcal{Y}^2 \ge 1 - \alpha^2/4\} = 1 - \pi(1-\alpha^2/4)\pi^{-1} = \alpha^2/4$. From this we get that $f_{\mathcal{L}}(\alpha) = \alpha/2$ for $0 \le \alpha \le 2$, and $f_{\mathcal{L}}(\alpha) = 0$ otherwise.

   (c) $\mathsf{E}[\mathcal{L}] = \displaystyle\int_0^2 \alpha \cdot \frac{\alpha}{2}\,d\alpha = \frac{4}{3}$