University of Illinois Spring 2008

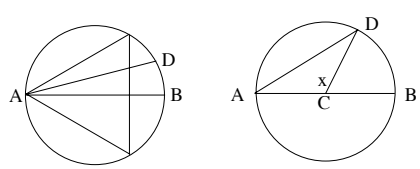
ECE 313: Solutions to Problem Set 10

1. (a) $\mathcal{I}$ can take on values in the range $(-I_0, \infty)$.
   (b) $F_{\mathcal{I}}(v) = 0$ for $v < -I_0$. For any $v > -I_0$,
   $F_{\mathcal{I}}(v) = P\{\mathcal{I} \le v\} = P\{I_0(\exp(\mathcal{V}) - 1) \le v\} = P\{\mathcal{V} \le \ln(1 + v/I_0)\} = F_{\mathcal{V}}(\ln(1 + v/I_0))$.
   (c) For $v > -I_0$,
   $$f_{\mathcal{I}}(v) = f_{\mathcal{V}}(\ln(1+v/I_0))\frac{1}{1+v/I_0} \times \frac{1}{I_0} = \frac{f_{\mathcal{V}}(\ln(1+v/I_0))}{v+I_0} = \begin{cases} \frac{I_0/2}{(v+I_0)^2}, & v \ge 0, \\ \frac{1}{2I_0}, & -I_0 < v < 0, \end{cases}.$$
   Note that the pdf has constant value $1/(2I_0)$ from $v = -I_0$ to $v = 0$.

2. (a) $\mathcal{Y}$ takes on values in $[0, 1]$ and hence $F_{\mathcal{Y}} = 0$ for $v < 0$, and $F_{\mathcal{Y}}(v) = 1$ for $v > 1$.
   For $0 \le v \le 1$, $F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \le v\} = P\{\mathcal{X}^2 \le v\} = P\{-\sqrt{v} \le \mathcal{X} \le \sqrt{v}\} = \sqrt{v}$.
   Hence $f_{\mathcal{Y}}(v) = \frac{1}{2\sqrt{v}}$ if $0 \le v \le 1$, and $f_{\mathcal{Y}}(v) = 0$, otherwise.
   (b) $\mathcal{Z}$ takes on values in $[-1, 1]$ and hence, $F_{\mathcal{Z}} = 0$ for $v < -1$, and $F_{\mathcal{Z}}(v) = 1$ for $v > 1$.
   For $0 \le v \le 1$, $F_{\mathcal{Z}}(v) = P\{\mathcal{Z} \le v\} = P\{g(\mathcal{X}) \le v\} = P\{\mathcal{X} \le \sqrt{v}\} = \frac{1}{2}[1 + \sqrt{v}]$.
   For $-1 \le v \le 0$, $F_{\mathcal{Z}}(v) = P\{\mathcal{Z} \le v\} = P\{g(\mathcal{X}) \le v\} = P\{\mathcal{X} \le \sqrt{-v}\} = \frac{1}{2}[1 - \sqrt{-v}]$.
   Hence, $f_{\mathcal{Z}}(v) = \frac{1}{4\sqrt{|v|}}$ if $0 \le |v| \le 1$, and $f_{\mathcal{Z}}(v) = 0$, otherwise. Note that the pdf is an even function, and approaches $+\infty$ as v approaches 0 from either side.

3. (a) The pmf of $\mathcal{Y}$ is $p_{\mathcal{Y}}(\alpha) = p_{\mathcal{Y}}(-\alpha) = \dfrac{1}{2}$.
   (b) $\mathsf{E}[\mathcal{Z}] = \displaystyle\int_0^\infty (u-\alpha)^2\phi(u)\,du + \int_{-\infty}^0 (u+\alpha)^2\phi(u)\,du = \int_{-\infty}^\infty (u^2+\alpha^2)\phi(u)\,du - 4\int_0^\infty \alpha u\phi(u)\,du$
   $= 1 + \alpha^2 - 2\sqrt{\dfrac{2}{\pi}}\alpha$ where we have used the facts that the standard Gaussian random variable has variance 1, the area under the pdf $\phi(u)$ is 1, and $\int_0^\infty u\exp(-u^2/2)\,du = 1$ (cf. Problem 5(b) of Problem Set 1) in arriving at the result. $\mathsf{E}[\mathcal{Z}]$ has minimum value $1 - \frac{2}{\pi}$ at $\alpha = \sqrt{2/\pi}$.
   (c) $p_{\mathcal{W}}(3) = p_{\mathcal{W}}(-3) = \Phi(-2.5) = 0.0062$. $p_{\mathcal{W}}(2) = p_{\mathcal{W}}(-2) = \Phi(2.5) - \Phi(1.5) = 0.0606$.
   $p_{\mathcal{W}}(1) = p_{\mathcal{W}}(-1) = \Phi(1.5) - \Phi(0.5) = 0.2417$. $p_{\mathcal{W}}(0) = \Phi(0.5) - \Phi(-0.5) = 0.3830$.
   (d) $\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0$ are *Bernoulli* random variables with parameters $p_2 = P\{\mathcal{W} < 0\} = 0.3085$, $p_1 = P\{\mathcal{W} \in \{-2, -1, 2, 3\}\} = 0.3691$, and $p_0 = P\{\mathcal{W} \in \{-3, -1, 1, 3\}\} = 0.4958$ respectively.

4. (a) $\mathcal{X}$ is uniformly distributed on $[0, 2\pi)$. From the diagram below, it should be obvious that the probability that the random chord is longer than the side of the inscribed equilateral triangle is $P\{2\pi/3 < \mathcal{X} < 4\pi/3\} = \frac{1}{3}$.



   (b) Since the circle has radius 1, an arc of length $\mathcal{X}$ subtends angle $\mathcal{X}$ at the center of the circle. Furthermore, the length $\mathcal{L}$ of the chord is $2\sin(\mathcal{X}/2)$, increasing from 0 when $\mathcal{X} = 0$ to 2 when $\mathcal{X} = \pi$ and decreasing back to 0 at $\mathcal{X} = 2\pi$. For any $x, 0 < x < 2$,
   $$F_{\mathcal{L}}(x) = P\{\mathcal{L} \le x\} = P\{2\sin(\mathcal{X}/2) \le x\} = 2{\cdot}P\{0 \le \mathcal{X} \le 2\arcsin(x/2)\} = \left(\frac{2}{\pi}\right)\arcsin\left(\frac{x}{2}\right)$$

. Hence,

$$f_{\mathcal{L}}(x) = \frac{d}{dx}F_{\mathcal{L}}(x) = \begin{cases} \dfrac{1}{\pi\sqrt{1-(x/2)^2}}, & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

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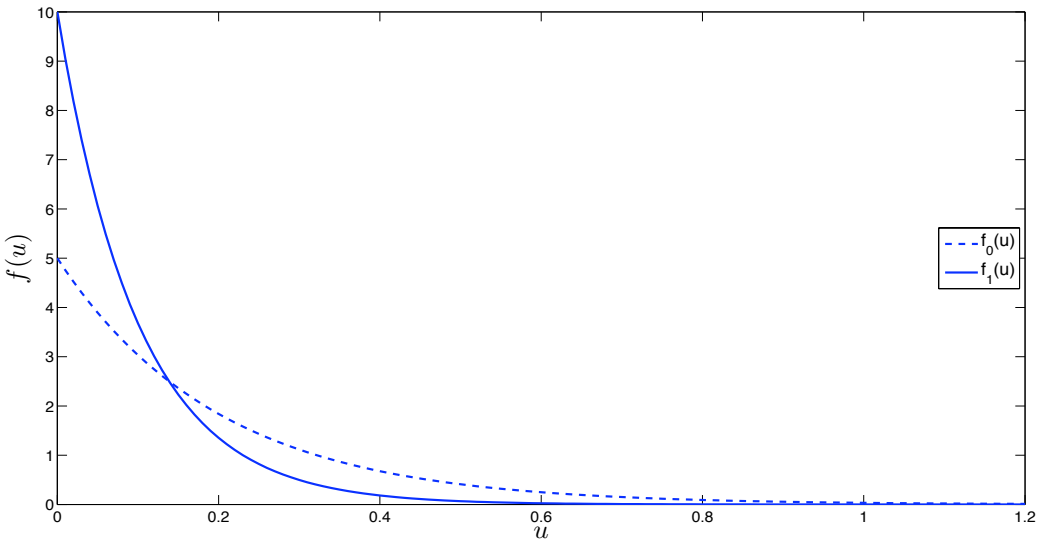
5. (a) The pdfs are as shown below.



(b) $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{10 \cdot \exp(-10u)}{5 \cdot \exp(-5u)} = 2 \cdot \exp(-5u)$
which has value 2 at $u = 0$ and decays away to 0 as $u \to \infty$. Note that $\Lambda(u) > 1$ for $u < 0.2 \ln 2$. Thus, the likelihood ratio test is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is *smaller* than the threshold $0.2 \ln 2$. Equivalently, $\Gamma_1 = (0, 0.2 \ln 2), \Gamma_0 = (0.2 \ln 2, \infty)$.

(c) $P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\, du = \int_0^{0.5 \ln 2} 5 \cdot \exp(-5u)\, du = -\exp(-5u)\Big|_0^{0.2 \ln 2} = -\frac{1}{2} - (-1) = \frac{1}{2}.$
$P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\, du = \int_{0.5 \ln 2}^{\infty} 10 \cdot \exp(-10u)\, du = -\exp(-10u)\Big|_{0.2 \ln 2}^{\infty}$
$= 0 - (-\exp(-2 \ln 2)) = \dfrac{1}{4}.$

(d) $\Lambda(u) = 2 \cdot \exp(-5u) > \dfrac{\pi_0}{\pi_1}$ for $u < 0.2 \ln\left(\dfrac{2\pi_1}{\pi_0}\right) = 0.2 \ln 2 + 0.2 \ln\left(\dfrac{\pi_1}{\pi_0}\right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is smaller than ξ. Note that $\xi < 0$ if $\pi_0 > 2\pi_1$, that is, if $\pi_0 > 2/3$.

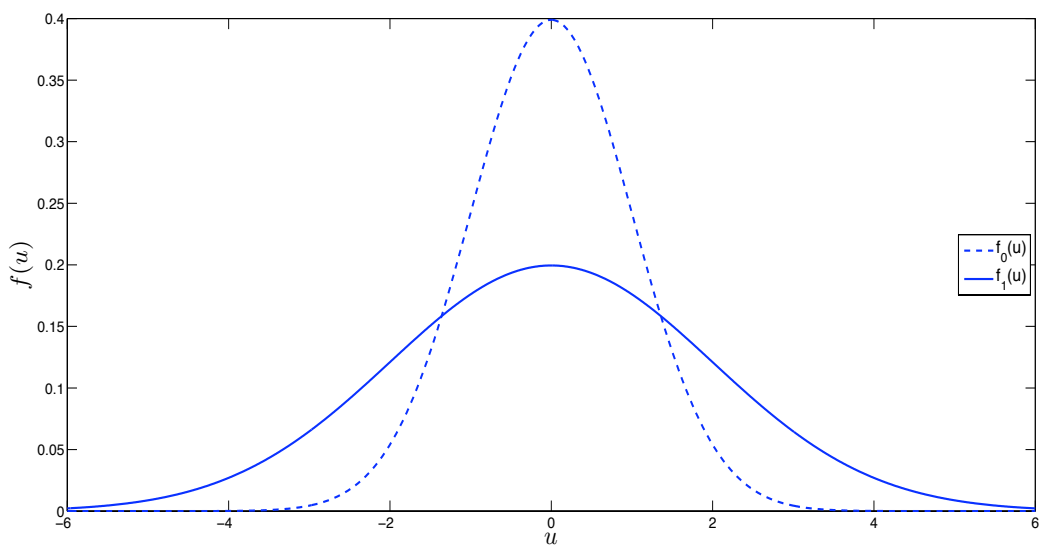
(e) If $\pi_0 = 1/3$, then $\xi = 0.2 \ln 4$. Hence,
$P_{\text{FA}} = \displaystyle\int_0^{0.2 \ln 4} 5 \cdot \exp(-5u)\, du = -\exp(-5u)\Big|_0^{0.2 \ln 4} = -\frac{1}{4} - (-1) = \frac{3}{4}.$
$P_{\text{MD}} = \displaystyle\int_{0.2 \ln 4}^{\infty} 10 \cdot \exp(-10u)\, du = -\exp(-10u)\Big|_{0.2 \ln 4}^{\infty} = 0 - (-\exp(-2 \ln 4)) = \frac{1}{16}.$
The average error probability thus is $\bar{P}_e = \dfrac{1}{3}P_{\text{FA}} + \dfrac{2}{3}P_{\text{MD}} = \dfrac{7}{24}$. Note that since $\pi_0 < \pi_1$,

the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

(f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $P_e = \pi_0$.

(g) When $\pi_0 > 2/3$, the threshold ξ is less than 0. Since $\mathcal{X}$ takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0. The average error probability is π_1, and since this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$. When $\pi_0 > 2/3$, it follows that $\pi_0 > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_0 \cdot (1/2) + \pi_1 \cdot (1/4) > 2\pi_1 \cdot (1/2) + \pi_1 \cdot (1/3) = 1.25\pi_1$.

6. (a) The pdfs are sketched below:



(b) $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{1/\sqrt{2\pi\sigma_1^2}\exp\left\{-u^2/2\sigma_1^2\right\}}{1/\sqrt{2\pi\sigma_0^2}\exp\left\{-u^2/2\sigma_0^2\right\}} = \dfrac{\sigma_0}{\sigma_1}\exp\left\{\dfrac{u^2}{2}\left[\dfrac{1}{\sigma_0^2} - \dfrac{1}{\sigma_1^2}\right]\right\}$

<u>ML Decision Rule</u>: Compare $\Lambda(u)$ to 1, or equivalently, compare $\log\Lambda(u)$ to 0:

$$\begin{aligned}
\log\Lambda(u) &\gtrless 0 \\
\log\left(\frac{\sigma_0}{\sigma_1}\exp\left\{\frac{u^2}{2}\left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right]\right\}\right) &\gtrless 0 \\
\log\frac{\sigma_0}{\sigma_1} + \frac{u^2}{2}\left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right] &\gtrless 0 \\
u^2 &\gtrless \frac{2\log\left(\frac{\sigma_1}{\sigma_0}\right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}} \\
|u| &\gtrless \sqrt{\frac{2\log\left(\frac{\sigma_1}{\sigma_0}\right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}} = \xi
\end{aligned}$$

So the ML decision rule becomes: **Choose H_1 if $|u| \geq \xi$, otherwise choose H_0.**

Bayes Decision Rule: Compare $\Lambda(u)$ to $\frac{\pi_0}{\pi_1}$, or equivalently, compare $\log \Lambda(u)$ to $\log \frac{\pi_0}{\pi_1}$:

$$\begin{aligned}
\log \Lambda(u) &\gtrless \log \frac{\pi_0}{\pi_1} \\
\log \left(\frac{\sigma_0}{\sigma_1} \exp \left\{ \frac{u^2}{2} \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right] \right\} \right) &\gtrless \log \frac{\pi_0}{\pi_1} \\
\log \frac{\sigma_0}{\sigma_1} + \frac{u^2}{2} \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right] &\gtrless \log \frac{\pi_0}{\pi_1} \\
u^2 &\gtrless \frac{2 \log \left(\frac{\sigma_1}{\sigma_0} \cdot \frac{\pi_0}{\pi_1} \right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}} \\
|u| &\gtrless \sqrt{\frac{2 \log \left(\frac{\sigma_1}{\sigma_0} \cdot \frac{\pi_0}{\pi_1} \right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}} = \gamma
\end{aligned}$$

So the Bayes decision rule becomes: **Choose H_1 if $|u| \geq \gamma$, otherwise choose H_0.**

(c) $\sigma_0^2 = 1, \sigma_1^2 = 4 \Rightarrow \xi = \sqrt{\frac{2 \log\left(\frac{2}{1}\right)}{\frac{1}{1} - \frac{1}{4}}} = 1.3596.$

Under H_0, $X \sim \mathcal{N}(0, 1)$.
$P_{\text{FA}} = P\{|X_0| \geq 1.3596\} = 1 - P\{|X_0| \leq 1.3596\} = 2\left(1 - \Phi(1.3596)\right) \approx 0.177$

Under H_1, $X \sim \mathcal{N}(0, 4)$.
$P_{\text{MD}} = P\{|X_1| \leq 1.3596\} = P\left\{-\frac{1.3596}{2} \leq \frac{X}{2} \leq \frac{1.3596}{2}\right\} = 2\Phi(0.6798) - 1 \approx 0.5034$