ECE 313: Solutions to Problem Set 9

1. (a) $\mathsf{E}[\mathcal{Y}] = \mathsf{E}[\mathcal{X}^2] = \int_{-1}^{+1} u^2 \left(\frac{1}{2}\right) du = \frac{1}{3}$. $\mathsf{E}[\mathcal{Y}^2] = \mathsf{E}[\mathcal{X}^4] = \int_{-1}^{+1} u^4 \left(\frac{1}{2}\right) du = \frac{1}{5}$.
   Hence, $\mathsf{var}(\mathcal{Y}) = \mathsf{E}[\mathcal{Y}^2] - (\mathsf{E}[\mathcal{Y}])^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$.

   (b) $\mathsf{E}[\mathcal{Z}] = \mathsf{E}[g(\mathcal{X})] = \int_{-1}^{0} -u^2 \left(\frac{1}{2}\right) du + \int_{0}^{+1} u^2 \left(\frac{1}{2}\right) du = -\frac{1}{6} + \frac{1}{6} = 0.$

   (c) $\mathcal{U}$ takes on integer values ≥ 1. $\mathcal{V} = \sin(\pi\mathcal{U}/2) = 0$ if $\mathcal{U}$ is even. On the other hand, if $\mathcal{U} = 4k+1$, then $\mathcal{V} = +1$ while if $\mathcal{U} = 4k-1$, then $\mathcal{V} = -1$. We get
   $$\mathsf{E}[\mathcal{V}] = \sum_{k=1}^{\infty} \sin(\pi k/2) \left(\frac{1}{2}\right)^k = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \cdots = \frac{1}{2}\left[1 - x + x^2 - \cdots\right] = \frac{1}{2(1+x)}$$
   where $x = \frac{1}{4} \quad \Rightarrow \mathsf{E}[\mathcal{V}] = \frac{2}{5}$.

2. (a) $\mathsf{E}[\mathcal{R}] = \int_0^1 u \cdot 2u\, du = \frac{2}{3}$. $\mathsf{E}[\mathcal{V}] = \int_0^1 \frac{4\pi}{3} u^3 \cdot 2u\, du = \frac{8\pi}{15}$. $\mathsf{E}[\mathcal{A}] = \int_0^1 4\pi u^2 \cdot 2u\, du = 2\pi$.
   The *average sphere* does not have average volume or average surface area. More generally, $\mathsf{E}[g(\mathcal{X})] \neq g(\mathsf{E}[\mathcal{X}])$.

   (b) For $0 < u < 1$, $u^n > u^{n+1}$.
   Hence, $\int_0^1 u^n \cdot f_{\mathcal{R}}(u)\, du = \mathsf{E}[\mathcal{R}^n] > \int_0^1 u^{n+1} \cdot f_{\mathcal{R}}(u)\, du = \mathsf{E}[\mathcal{R}^{n+1}]$.

3. I'm considering, I'm considering!

   (a) The number of arrivals in $(0,4]$ is $N(0,4]$, a Poisson random variable with parameter 4λ. Hence, $\mathsf{E}[N(0,4]] = 4\lambda$.

   (b) The event $\{N(0,3] = 3\} \cap \{N(2,6] = 0\}$ is the same as $\{N(0,2] = 3\} \cap \{N(2,6] = 0\}$ and thus $P[\{N(0,3] = 3\} \cap \{N(2,6] = 0\}] = P[\{N(0,2] = 3\} \cap \{N(2,6] = 0\}]$. We get
   $$P[\{N(0,3] = 3\} \cap \{N(2,6] = 0\}] = \frac{(2\lambda)^3}{3!}\exp(-2\lambda) \times \exp(-4\lambda) = \frac{4\lambda^3}{3}\exp(-6\lambda)$$
   by independence of Poisson variables on disjoint intervals of time.

   (c) The number of arrivals in $(0,6]$ is $N(0,6]$, a Poisson random variable with parameter 6λ. If it is observed that $N(0,6] = k$, then the maximum-likelihood estimate of the Poisson parameter 6λ is k (cf. Problem 2(c) of Problem Set 5), and hence the maximum-likelihood estimate of the arrival rate is $k/6$ = # of arrivals/length of interval which is exactly what one would expect.

   (d) $P\{N(0,t) \geq 1\} = 1 - P\{N(0,t) = 0\} = 1 - \exp(-\lambda t) = 1 - 2^{-t}$ when $\lambda = \ln 2$.

4. (a) The inter-arrival time in a Poisson process with arrival rate λ (time between two successive chalk breaks on this instance) is an exponential random variable with parameter λ. Hence, the average length of time between successive chalk-breaks is the mean of this exponential random variable, which is $\lambda^{-1} = 10$ minutes.

   (b) The number of times that Todd breaks the chalk during a 50 minute lecture is a Poisson random variable $N(0,50]$ with parameter $\lambda \times 50 = 5$ and mean value $\mathsf{E}[N(0,50]] = 5$.

(c) From part (b), we get that $P\{N(0,50]=6\} = \dfrac{5^6}{6!}\exp(-5)$. Now, for $0 \le k \le 6$,

$$\begin{aligned}
P\left\{\{N(0,25]=k\}\,|\,\{N(0,50]=6\}\right\} &= \frac{P\left\{\{N(0,25]=k\}\cap\{N(0,50]=6\}\right\}}{P\{N(0,50]=6\}\}}\\
&= \frac{P\left\{\{N(0,25]=k\}\cap\{N(25,50]=6-k\}\right\}}{P\{N(0,50]=6\}\}}\\
&= \frac{P\{N(0,25]=k\}P\{N(25,50]=6-k\}\}}{P\{N(0,50]=6\}\}}\\
&= \frac{\frac{(2.5)^k}{k!}\exp(-2.5)\times\frac{(2.5)^{6-k}}{(6-k)!}\exp(-2.5)}{\frac{5^6}{6!}\exp(-5)}\\
&= \binom{6}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{6-k}
\end{aligned}$$

Thus, the *conditional* pmf of $N(0,25]$ given that $\{N(0,50]=6\}$ is a *binomial* pmf with parameters $(6,\frac{1}{2})$ and hence the expected value is 3.

5. (a) $P\{\mathcal{X}<0\} = \Phi\left(\dfrac{0-(-10)}{2}\right) = \Phi(5) = 1-Q(5)$.

   (b) $P\{-10 < \mathcal{X} < 5\} = \Phi\left(\frac{5-(-10)}{2}\right) - \Phi\left(\frac{-10-(-10)}{2}\right) = \Phi(7.5)-\Phi(0) = \Phi(7.5)-\frac{1}{2} = \frac{1}{2}-Q(7.5)$.

   (c) $P\{|\mathcal{X}|\ge 5\} = P\{\mathcal{X}\le -5\} + P\{\mathcal{X}\ge 5\} = \Phi\left(\frac{-5-(-10)}{2}\right) + 1 - \Phi\left(\frac{5-(-10)}{2}\right) = \Phi(2.5) + 1-\Phi(7.5) = 1-Q(2.5)+Q(7.5)$.

   (d) $P\{\mathcal{X}^2-3\mathcal{X}+2>0\} = P\{(\mathcal{X}-1)(\mathcal{X}-2)>0\} = P\{\mathcal{X}<1\}+P\{\mathcal{X}>2\} = \Phi\left(\frac{1-(-10)}{2}\right) + 1-\Phi\left(\frac{2-(-10)}{2}\right) = \Phi(5.5)+1-\Phi(6) = 1-Q(5.5)+Q(6)$.

6. Let $\mathcal{X}\sim\mathcal{N}(0.9, 0.003^2)$ denote the width (in microns) of the trace.

   (a) $\{\mathcal{X}<0.9-0.005\}$ or $\{\mathcal{X}>0.9+0.005\}$ for a trace to be deemed defective. $P\{|\mathcal{X}-0.9|>0.005\} = 2\Phi(-0.005/0.003) = 2\Phi(-1.666\ldots) = 2Q(1.666\ldots)\approx 0.095$.

   (b) We need to find the maximum value of σ such that $2Q(0.005/\sigma)\le 10^{-2}$. Since $Q(2.575)\approx 0.005$, we get that $0.0005/\sigma > 2.575$, that is, $\sigma \le 0.005/2.575 \approx 0.00194$.

7. $Y(f) = H(f)X(f) = \mathrm{rect}(f/2)\exp(-\pi f^2)$, and $y(0) = \displaystyle\int_{-\infty}^{\infty} Y(f)\,df = \int_{-1}^{1}\exp(-\pi f^2)\,df$. But, the *integrand* is the Gaussian pdf $\mathcal{N}(0,(2\pi)^{-1})$. Hence, $y(0) = \Phi(\sqrt{2\pi})-\Phi(-\sqrt{2\pi}) = 2\Phi(\sqrt{2\pi})-1\approx 0.9826$