University of Illinois Spring 2008

ECE 313: Solutions to Problem Set 7

1. (a)

$$\begin{aligned}
P(D) &= P((A\cap B)\cup(B\cap C)\cup(A\cap C))\\
&= P(A\cap B)+P(A\cap C)+P(B\cap C)-2P(A\cap B\cap C)\\
&= P(A)P(B)+P(A)P(C)+P(B)P(C)-2P(A)P(B)P(C) \quad \text{by independence}\\
&= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)-2\times\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\\
&= \frac{13}{24}
\end{aligned}$$

(b) $P(A^c|D) = \dfrac{P(A^c\cap D)}{P(D)} = \dfrac{P(A^c\cap B\cap C)}{13/24} = \dfrac{(2/3)\times(1/2)\times(3/4)}{13/24} = \dfrac{6}{13}$

2. (a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e., the event $A\cap(B\cup C\cup D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e., the event $\{A^c\cap B\cap C\cap D\}$ occurs. Note that these are disjoint events (Why?). Now, $(B\cup C\cup D)^c = B^c\cap C^c\cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows: $A = A\cap(B\cup C\cup D)\cup(A\cap(B^c\cap C^c\cap D^c))$. Hence, we see that

$$P(A\cap(B\cup C\cup D)) = P(A)-P(A\cap B^c\cap C^c\cap D^c) = P(A)-P(A)P(B^c)P(C^c)(D^c)$$

by independence. Since $P(A^c\cap B\cap C\cap D) = P(A^c)P(B)P(C)P(D)$ by independence, we get

$$\begin{aligned}
P(\text{bill passes}) &= P(A\cap(B\cup C\cup D))+P(A^c\cap B\cap C\cap D)\\
&= P(A)-P(A)P(B^c)P(C^c)P(D^c)+P(A^c)P(B)P(C)P(D)\\
&= 0.9\times[1-0.4\times0.5\times0.8]+0.1\times0.6\times0.5\times0.2\\
&= 0.756+0.006 = 0.762
\end{aligned}$$

(b) The Conservative Republicans must be in favor for the motion to override to pass. Furthermore, since politics makes strange bedfellows, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence,

$$\begin{aligned}
P(\text{veto overridden}) &= P(E\cap(H\cup(F\cap G)))\\
&= P(E)\times P(H\cup(F\cap G)) \quad \text{by independence of } E \text{ and } H\cup(F\cap G)\\
&= P(E)\times[P(H)+P(F\cap G)-P(F\cap G\cap H)]\\
&= P(E)\times[P(H)+P(F)P(G)-P(F)P(G)(H)] \quad \text{independence again!}\\
&= 0.99\times[0.1+0.4\times0.6-0.6\times0.4\times0.1]\\
&= 0.31284
\end{aligned}$$

3. (a) For the single gigantic car, each part fails with probability p. All the N parts of a given type must fail in order for the car to fail, and this occurs with probability p^N, and so the probability that at least one part of a given type is working is $1-p^N$. The probability that at least one part of each of the M types is working is $(1-p^N)^M$, and hence the probability of system failure is $1-(1-p^N)^M$.

On the other hand, if we have N cars each with M parts, then the probability that at least one part fails in a car is $1-(1-p)^M$. This is also the probability that the car itself has failed. But, we have N cars, and thus the probability that at least one car is in working condition (i.e. that not all have failed) is $(1-(1-p)^M)^N$.

(b) For the single gigantic car, we want $1-((1-0.2^N)^5 < 10^{-3}$ which is achieved by choosing $N \geq 6$. For the N small cars, we want $(1-(1-0.2)^5)^N < 10^{-3}$ which is achieved by choosing $N \geq 18$. Thus, a gigantic car with 6 engines, 6 transmissions, 6 brakes etc provides more reliable transmission than having 17 different compact cars.

(c) If $M = 1000$, we get that $N \geq 9$ for the gigantic car while typical calculators blow up on the calculation for the number of small cars needed. The Unix high-precision utility *bc* gives that $1-(1-0.2)^{1000} = 0.99\ldots 9986160344\ldots$ where there are eightyseven 9's preceding the 8616, and the number of cars needed is astronomical. Parking, of course, is impossible, but keep in mind that a gigantic car with 9 engines etc will also be very difficult to park!

Moral: It is better to replicate parts than to replicate systems.

4. (a) $\Lambda(k) = \dfrac{p_1(k)}{p_0(k)} = \dfrac{(\ln 27)^k \exp(-\ln 27)/k!}{(\ln 9)^k \exp(-\ln 9)/k!} = \left(\dfrac{\ln 27}{\ln 9}\right)^k \dfrac{9}{27} = \left(\dfrac{3\ln 3}{2\ln 3}\right)^k \dfrac{1}{3} = \dfrac{3^{k-1}}{2^k}$

(b) Evaluating $\Lambda(k)$ for $k = 0, 1, 2, \ldots$, we get that the decision is in favor of H_1 if $\mathcal{X} \geq 3$.

(c) $P_{\text{FA}} = \displaystyle\sum_{k=3}^{\infty} \frac{(\ln 9)^k}{k!} \exp(-\ln 9) = 1 - \frac{1}{9}\sum_{k=0}^{2} \frac{(\ln 9)^k}{k!} = 1 - \frac{1}{9}\left[1 + \ln 9 + \frac{(\ln 9)^2}{2!}\right]$

$P_{\text{MD}} = \displaystyle\sum_{k=0}^{2} \frac{(\ln 27)^k}{k!} \exp(-\ln 27) = \frac{1}{27}\left[1 + \ln 27 + \frac{(\ln 27)^2}{2!}\right]$

5. (a) If $\mathcal{X} = n$, the likelihood ratio has value

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1}{p_0}\left(\frac{1-p_1}{1-p_0}\right)^{n-1} > 1 \text{ if } (n-1)\ln\left(\frac{1-p_1}{1-p_0}\right) > \ln\left(\frac{p_0}{p_1}\right)$$

Since $p_1 < p_0$, we have that $1 - p_1 > 1 - p_0$ and $\ln((1-p_1)/(1-p_0)) > 0$. Therefore, the maximum likelihood decision rule is

$$\text{“Decide that } \mathsf{H}_1 \text{ is the true hypothesis if } \mathcal{X} > 1 + \frac{\ln\left(\frac{p_0}{p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}\text{”}$$

(b) The minimum-error-probability (MEP) decision rule decides that H_1 is the true hypothesis if the likelihood ratio exceeds the threshold π_0/π_1. Now $\Lambda(1) = p_1/p_0 < 1$. Since $1 - p_1 > 1 - p_0$, we see that

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1(1-p_1)^{n-2}}{p_0(1-p_0)^{n-2}}\left(\frac{1-p_1}{1-p_0}\right) = \Lambda(n-1)\left(\frac{1-p_1}{1-p_0}\right) > \Lambda(n-1),$$

and so $\Lambda(1) = p_1/p_0$ is the smallest value of the likelihood ratio. It follows that if $\pi_0/\pi_1 = \pi_0/(1-\pi_0) < p_1/p_0$, that is, if $\pi_0 < p_1/(p_0 + p_1)$, the MEP decision rule will always decide that H_1 is the true hypothesis regardless of the observed value of $\mathcal{X}$.

On the other hand, since $\Lambda(n)$ increases monotonically without bound as n increases, there is *no* value of $\pi_0 < 1$ for which π_0/π_1 can be guaranteed to be larger than the likelihood ratio no matter what value $\mathcal{X}$ takes on.

6. (a) The maximum-likelihood decision rule is indicated by the boldface entries in the likelihood matrix shown below.

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0: excellent	0.02	0.08	0.15	**0.75**
H_1: good	0.1	0.15	**0.6**	0.15
H_2: average	**0.2**	**0.65**	0.1	0.05

(b) P(excellent student is labeled as good) $= P(\mathcal{X} = 9|\mathsf{H}_0) = 0.15$.
P(excellent student is labeled as average) $= P(\{\mathcal{X} = 6\} \cup \{\mathcal{X} = 3\}|\mathsf{H}_0) = 0.02 + 0.08 = 0.1$.
P(average student is labeled as good or excellent) $= P(\{\mathcal{X} = 9\} \cup \{\mathcal{X} = 12\}|\mathsf{H}_2) = 0.15$.

(c) The conditional error probabilities of the maximum-likelihood decision rule are $P(E|\mathsf{H}_0) = 0.25$, $P(E|\mathsf{H}_1) = 0.4$, $P(E|\mathsf{H}_02) = 0.15$. Hence, the error probability is

$$P(E) = P(E|\mathsf{H}_0)P(\mathsf{H}_0) + P(E|\mathsf{H}_1)P(\mathsf{H}_1) + P(E|\mathsf{H}_2)P(\mathsf{H}_2) = 0.05 + 0.22 + .0375 = 0.3075.$$

(d) The joint probability matrix is as shown below together with the MAP decision rule.

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0: excellent	0.004	0.016	0.03	**0.15**
H_1: good	**0.055**	0.0825	**0.33**	0.0825
H_2: average	0.05	**0.1625**	0.025	0.0125

$P(E) = 1 - (0.15 + 0.33 + 0.1625 + 0.055) = 0.3025$ which is slightly smaller than that of the maximum-likelihood rule. But note that students getting D's are classified as good while students getting C's are classified as average. Holy capricious grading complaint, Batman!

(e) Now the joint probability matrix looks as shown below, and all students are classified as excellent regardless of their grade on the exam!

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0: excellent	**0.019**	**0.076**	**0.1425**	**0.7125**
H_1: good	0.005	0.0075	0.03	0.0075
H_2: average	0	0	0	0

Obviously, there is no need for examinations at the Lake Wobegon campus since the results are ignored anyway.

Noncredit exercise: Write a letter to the Governor asking him to suggest to the University that the Lake Wobegon approach be adopted as a cost-cutting measure ...