University of Illinois Spring 2008

ECE 313: Solutions to Problem Set 6

1. Since an event D can be *partitioned* into two disjoint subsets DE and DE^c, we have that $P(D) = P(DE) + P(DE^c)$. This fact is used several times below.

   (a) $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(ABC) + P(ABC^c)}{P(B)} = \dfrac{P(ABC)P(BC)}{P(BC)P(B)} + \dfrac{P(ABC^c)P(BC^c)}{P(BC^c)P(B)}$
   $= P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$.

   (b) $P(A|BC) = \dfrac{P(ABC)}{P(BC)} = \dfrac{P(ABC)}{P(ABC) + P(A^cBC)} = \dfrac{\frac{P(ABC)}{P(AB)} \cdot \frac{P(AB)}{P(B)}}{\frac{P(ABC)}{P(AB)} \cdot \frac{P(AB)}{P(B)} + \frac{P(A^cBC)}{P(A^cB)} \cdot \frac{P(A^cB)}{P(B)}}$
   $= \dfrac{P(C|AB)P(A|B)}{P(C|AB)P(A|B) + P(C|A^cB)P(A^c|B)}$.

   (c) $P(A|BC) = \dfrac{P(ABC)}{P(BC)} = \dfrac{P(C|AB)P(AB)}{P(C|B)P(B)} = \dfrac{P(C|AB)P(A|B)P(B)}{P(C|B)P(B)} = \dfrac{P(C|AB)P(A|B)}{P(C|B)}$

2. (a) This is sampling with replacement and we have that $P(R_1) = P(R_2) = \frac{r}{r+g}$.

   (b) $P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|R_1^c)P(R_1^c) = \dfrac{r+c}{r+c+g} \times \dfrac{r}{r+g} + \dfrac{r}{r+c+g} \times \dfrac{g}{r+g}$
   $= \dfrac{r}{r+g}$ just as before, *and independent of the value of* c!!

   (c) $P\{r+c \text{ red balls}|R_2\} = \dfrac{P(R_1 \cap R_2)}{P(R_2)} = \dfrac{P(R_2|R_1)P(R_1)}{P(R_2)} = P(R_2|R_1) = \dfrac{r+c}{r+c+g}$

3. (a) Since Todd is up-to-date at the beginning of the semester, he is up-to-date after one week with probability $P(U_1) = 0.8$, and behind with probability $P(B_1) = 0.2$.
   Yes, $P(U_1) + P(B_1) = 1$ because they are mutually exclusive events that partition the sample space, that is, $B_1 = U_1^c$.

   (b) $P(U_2) = P(U_2|U_1)P(U_1) + P(U_2|B_1)P(B_1) = 0.8 \times 0.8 + 0.2 \times 0.6 = 0.76$
   $P(B_2) = P(B_2|U_1)P(U_1) + P(B_2|B_1)P(B_1) = 0.2 \times 0.8 + 0.4 \times 0.2 = 0.24$
   Yes, $P(U_2) + P(B_2) = 1$ because they are mutually exclusive events that partition the sample space.

   (c) $P(U_{i+1}|U_i) = 0.8$ and $P(B_{i+1}|B_i) = 0.4$.

   (d) $P(U_{i+1}) = P(U_{i+1}|U_i)P(U_i) + P(U_{i+1}|B_i)P(B_i) = 0.8P(U_i) + 0.6P(B_i)$
   $P(B_{i+1}) = P(B_{i+1}|U_i)P(U_i) + P(B_{i+1}|B_i)P(B_i) = 0.2P(U_i) + 0.4P(B_i)$

   $$\begin{bmatrix} P(U_{i+1}) \\ P(B_{i+1}) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} P(U_i) \\ P(B_i) \end{bmatrix}$$

   $P(U_{i+1}) + P(B_{i+1}) = 0.8P(U_i) + 0.6P(B_i) + 0.2P(U_i) + 0.4P(B_i) = P(U_i) + P(B_i) = 1$.

   (e) $P(B_3) = 0.2P(U_2) + 0.4P(B_2) = 0.2 \times 0.76 + 0.4 \times 0.24 = 0.248 \Rightarrow P(U_3) = 0.752$.
   $P(B_2|B_3) = \dfrac{P(B_3|B_2)P(B_2)}{P(B_3)} = \dfrac{0.4 \times 0.24}{0.248} = 0.387\ldots$

   (f) $P(U_{i+1}) = 0.8P(U_i) + 0.6P(B_i) = 0.8P(U_i) + 0.6[1 - P(U_i)] = 0.6 + 0.2P(U_i)$. The solution to this difference equation is of the form $P(U_i) = A\alpha^i + B$ where from $P(U_0) = 1$ we get that $A + B = 1$. More generally, $A\alpha^{i+1} + B = 0.6 + 0.2(A\alpha^i + B)$ must hold for all i, that is, $A\alpha^i(\alpha - 0.2) = 0.6 - 0.8B$ must hold for all i. This can only happen if $\alpha = 0.2$ and $B = 0.75 \Rightarrow P(U_i) = 0.25(0.2)^i + 0.75$ giving us $P(U_0) = 1, P(U_1) = 0.8, P(U_2) = 0.76, \cdots$ and $\lim_{i\to\infty} P(U_i) = \lim_{i\to\infty} 0.25(0.2)^i + 0.75 = 0.75$

4. (a) $P(F_1) = P(F_1|E_1)P(E_1) + P(F_1|E_2)P(E_2) + P(F_1|E_3)P(E_3) + P(F_1|E_4)P(E_4)$ $= (0.9)(0.4) + (0)(0.3) + (0)(0.2) + (0)(0.1) = 0.36$

(b) If the MS was found on the first page, it implies that the MS is located in cell 1 $\Rightarrow$ $P(E_1|F_1) = 1$.

(c) No matter which cell the MS is located in, the first time the MS is paged in the cell in which it is located, it will be missed with probability 0.1. Given that the MS is missed in the first round, it will be missed again in the second round with conditional probability 0.1. So $P(G_8) = (0.1)(0.1) = 0.01$.

(d) Now $P(F_2|G_1) = \frac{P(F_2G_1}{P(G_1)}$. By the law of total probability with the partition consisting of events E_1, E_2, E_3, E_4,

$$\begin{aligned} P(G_1) &= P(G_1|E_1)P(E_1) + P(G_1|E_2)P(E_2) + P(G_1|E_3)P(E_3) + P(G_1|E_4)P(E_4) \\ &= (0.1)(0.4) + (1)(0.3) + (1)(0.2) + (1)(0.1) = 0.64 \end{aligned}$$

and

$$\begin{aligned} P(F_2G_1) &= P(F_2G_1|E_1)P(E_1) + P(F_2G_1|E_2)P(E_2) + P(F_2G_1|E_3)P(E_3) \\ &\quad +P(F_2G_1|E_4)P(E_4) \\ &= (0)(0.4) + (0.9)(0.3) + (0)(0.2) + (0)(0.1) = 0.27 \end{aligned}$$

Thus, $P(F_2|G_1) = \dfrac{0.27}{0.64} \approx 0.4219$.

(e) We are seeking $P(F_4|G_3) = \dfrac{P(F_4G_3)}{P(G_3)}$,where

$$\begin{aligned} P(G_3) &= P(G_3|E_1)P(E_1) + P(G_3|E_2)P(E_2) + P(G_3|E_3)P(E_3) \\ &\quad +P(G_3|E_4)P(E_4) \\ &= (0.1)(0.4) + (0.1)(0.3) + (0.1)(0.2) + (1)(0.1) = 0.19 \end{aligned}$$

and

$$\begin{aligned} P(F_4G_3) &= P(F_4G_3|E_1)P(E_1) + P(F_4G_3|E_2)P(E_2) + P(F_4G_3|E_3)P(E_3) \\ &\quad +P(F_4G_3|E_4)P(E_4) \\ &= (0)(0.4) + (0)(0.3) + (0)(0.2) + (0.9)(0.1) = 0.09 \end{aligned}$$

Thus, $P(F_4|G_3) = \dfrac{0.09}{0.19} \approx 0.4737$.

(f) We are seeking $P(F_8|G_7) = \dfrac{P(F_8G_7)}{P(G_7)}$,where

$$\begin{aligned} P(G_7) &= P(G_7|E_1)P(E_1) + P(G_7|E_2)P(E_2) + P(G_7|E_3)P(E_3) \\ &\quad +P(G_7|E_4)P(E_4) \\ &= (0.01)(0.4) + (0.01)(0.3) + (0.01)(0.2) + (0.1)(0.1) = 0.019 \end{aligned}$$

and

$$\begin{aligned} P(F_8G_7) &= P(F_8G_7|E_1)P(E_1) + P(F_8G_7|E_2)P(E_2) + P(F_8G_7|E_3)P(E_3) \\ &\quad +P(F_8G_7|E_4)P(E_4) \\ &= (0)(0.4) + (0)(0.3) + (0)(0.2) + (0.09)(0.1) = 0.009 \end{aligned}$$

Thus, $P(F_8|G_7) = \dfrac{0.009}{0.019} \approx 0.4737$.

Note: An explanation for why the answer to (d) is the same as the answer to (c) is that if the MS is not found after the first round of four pages, the conditional distribution of which cell the MS is located in is the same as the original distribution.

5. Let A denote the event that your initial choice is the curtain concealing the car and B the event that your final choice is the curtain concealing the car. Clearly $P(A) = \frac{1}{3}$. Now the value of $P(B|A)$ and $P(B|A^c)$ (and therefore $P(B)$) depends on your *strategy* in response to Monty's blandishments.

   (a) Suppose that you always switch. Then, obviously $P(B|A) = 0$ since you chose the curtain with the car initially and are now choosing the other curtain (which definitely hides a goat.) Also, $P(B|A^c) = 1$ since you had a goat initially, and you know where the other goat is. Thus, $P(B) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$.

   (b) If you always stay put, then $P(B|A) = 1$ and $P(B|A^c) = 0$ leading to $P(B) = \frac{1}{3}$.

   (c) If, say, you toss a fair coin (independently!) to decide whether to stay put (H) or switch (T), then $P(B|A,H) = 1, P(B|A,T) = 0$ and therefore $P(B|A) = P(B|A,H)P(H) + P(B|A,T)P(T) = \frac{1}{2}$. Similarly $P(B|A^c) = \frac{1}{2}$. Monty is correct in his assertion. Besides, he said it on national TV! He wouldn't lie to you on national TV, would he?

   Many students are puzzled by the results of this problem. Consider that you have a $\frac{1}{3}$ chance of choosing the curtain with the car in the first place. If you never switch, you win the car with probability $\frac{1}{3}$. The probability that one of the *other* two curtains is concealing the car is $\frac{2}{3}$. Monty in essence is asking "Would you rather have what's behind both those two other two curtains except you don't *have* to take any goats home with you unless you really want to, and oh, by the way, here is one goat which is not telling you anything new since you know that there is at least one goat behind the other two curtains."

   Exercise: What would be your chances of winning if you tossed a biased coin?

   (d) It makes no difference whether you or your friend chooses first; you have equal probability of choosing the curtain with the car. He just happens to have been unlucky. In this game, but you should stay put because the chances are $\frac{2}{3}$ that the car is behind one of the two curtains picked by you and your friend. He's already gotten the goat, and so you get the car with probability $\frac{2}{3}$.

6. (a) Note that each row of the matrix (call it A) is just the *conditional* pmf of $\mathcal{Y}$ conditioned on the value of $\mathcal{X}$. Let $\vec{P}_{\mathcal{X}} = [0.5, 0.25, 0.25]$ be the *pmf vector* for $\mathcal{X}$. Then, $\vec{P}_{\mathcal{Y}} = \vec{P}_{\mathcal{X}} A$. More specifically, $P\{\mathcal{Y} = j\} = \sum_{i=1}^{3} P\{\mathcal{Y} = j | \mathcal{X} = i\} P\{\mathcal{X} = i\}$

$$= \begin{cases} 0.8 \times 0.5 + 0.05 \times 0.25 + 0.15 \times 0.25 & = 0.45 \quad \text{for } j = 1 \\ 0.1 \times 0.5 + 0.9 \times 0.25 + 0.05 \times 0.25 & = 0.2875 \quad \text{for } j = 2 \\ 0.1 \times 0.5 + 0.05 \times 0.25 + 0.8 \times 0.25 & = 0.2625 \quad \text{for } j = 3 \end{cases}$$

   Sanity check: the three probabilities sum to 1.

   (b) $$P\{\mathcal{X} = i | \mathcal{Y} = 3\} = \frac{P\{\mathcal{Y} = 3 | \mathcal{X} = i\} P\{\mathcal{X} = i\}}{P\{\mathcal{Y} = 3\}} = \begin{cases} \frac{0.1 \times 0.5}{0.2625} & = \frac{4}{21} \quad \text{for } i = 1, \\ \frac{0.05 \times 0.25}{0.2625} & = \frac{1}{21} \quad \text{for } i = 2, \\ \frac{0.8 \times 0.25}{0.2625} & = \frac{16}{21} \quad \text{for } i = 3. \end{cases}$$

   This is the conditional pmf of $\mathcal{X}$ given that $\mathcal{Y}$ was observed to have value 3. Note that the sum of the three probabilities is 1, as it should be for a valid pmf.