University of Illinois Spring 2008

ECE 313: Solutions to Problem Set 5

1. (a) $\mathcal{X}$ denotes a binomial random variable with parameters (N, p). It counts the number of occurrences of an event A of probability p on N independent trials. $\mathcal{Y} = N - \mathcal{X}$ counts the number of occurrences of A^c, an event of probability $1-p$, on the N independent trials and is thus a binomial random variable with parameters $(N, 1-p)$.

   (b) $P\{\mathcal{X} \text{ is even}\} = \binom{N}{0}p^0(1-p)^N + \binom{N}{2}p^2(1-p)^{N-2} + \binom{N}{4}p^4(1-p)^{N-4} + \cdots$
   Now, from the binomial theorem (Ross, page 8) we get that
   $(x+y)^N + (-x+y)^N = 2\left[\binom{N}{0}x^0y^N + \binom{N}{2}x^2y^{N-2} + \binom{N}{4}p^4(1-p)^{N-4} + \cdots\right]$, and so, setting $x = p,\ y = 1-p$, we get $P\{\mathcal{X} \text{ is even}\} = \frac{1}{2}\left[(p+1-p)^N + (-p+1-p)^N\right] = \frac{1}{2}\left[1 + (1-2p)^N\right]$. Notice that the probability is $1/2$ for $p = 1/2$, 1 for $p = 0$, and 1 (or 0) for $p = 1$ according as N is even or odd.

2. (a) $P\{\mathcal{Y} \text{ is even}\} = P\{\mathcal{Y} = 0\} + P\{\mathcal{Y} = 2\} + P\{\mathcal{Y} = 4\} + \cdots = \exp(-\lambda)\left[1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right]$
   $= \exp(-\lambda)\cosh(\lambda)$ by recognizing the series in square brackets.

   (b) $\frac{1}{2}\left[1 + (1-2p)^N\right] = \frac{1}{2}\left[1 + \left(1 - \frac{2\lambda}{N}\right)^N\right] \to \frac{1}{2}[1 + \exp(-2\lambda)] = \exp(-\lambda)\frac{\exp(\lambda)+\exp(-\lambda)}{2}$
   $= \exp(-\lambda)\cosh(\lambda)$.

   (c) For fixed $\mathcal{Y} = k$, the likelihood function (parameterized by θ) is $L(\theta; k) = \dfrac{\theta^k e^{-\theta}}{k!}$. To find the ML estimate of λ, we find $\hat{\theta}$ that maximizes $L(\theta; k)$. Setting the derivative of $L(\theta; k)$ equal to zero, we get

$$\frac{\partial L(\theta;k)}{\partial \theta} = \frac{1}{k!}\left[k\theta^{k-1}e^{-\theta} + \theta^k e^{-\theta}(-1)\right] = \frac{\theta^{k-1}e^{-\theta}}{k!}[k-\theta] = 0 \quad \Rightarrow \hat{\theta}_{ML} = k.$$

3. (a) On average, $\mathsf{E}[\mathcal{X}] = 105 \times 0.9 = 94.5$ passengers show up for the flight.

   (b)

$$\begin{aligned}
P\{\mathcal{X} \le 100\} &= 1 - P\{\mathcal{X} > 100\} = 1 - \sum_{k=101}^{105} P\{\mathcal{X} = k\} \\
&= 1 - \binom{105}{101}(0.9)^{101}(0.1)^4 - \binom{105}{102}(0.9)^{102}(0.1)^3 - \binom{105}{103}(0.9)^{103}(0.1)^2 \\
&\quad - \binom{105}{104}(0.9)^{104}(0.1)^1 - \binom{105}{105}(0.9)^{105}(0.1)^0 \\
&= 1 - \binom{105}{4}(0.9)^{101}(0.1)^4 - \binom{105}{3}(0.9)^{102}(0.1)^3 - \binom{105}{2}(0.9)^{103}(0.1)^2 \\
&\quad - \binom{105}{1}(0.9)^{104}(0.1)^1 - \binom{105}{0}(0.9)^{105}(0.1)^0 \\
&= 0.9832\ldots
\end{aligned}$$

   (c) We saw earlier that if $\mathcal{X}$ is a binomial random variable with parameters (n, p), then $\mathcal{Y} = n - \mathcal{X}$ is a binomial random variable with parameters $(n, 1-p)$.

   (d) $P\{\mathcal{Y} \ge 5\} = 1 - P\{\mathcal{Y} = 0\} - P\{\mathcal{Y} = 1\} - P\{\mathcal{Y} = 2\} - P\{\mathcal{Y} = 3\} - P\{\mathcal{Y} = 4\} = 1 - \exp(-10.5)\left[1 + \frac{10.5}{1!} + \frac{(10.5)^2}{2!} + \frac{(10.5)^3}{3!} + \frac{(10.5)^4}{4!}\right] = 0.9789\ldots.$

4. (a) At least two boxes of Cornies must be bought.

(b) For $k \geq 2$, $P\{\mathcal{X} = k\} = P(HH\cdots HB) + P(BB\cdots BH) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1}$.

$$\begin{aligned}
\mathsf{E}[\mathcal{X}] &= \sum_{k=2}^{\infty} k \cdot \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1}\right] \\
&= \sum_{k=1}^{\infty} k \cdot \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-1}\right] - \frac{1}{3} - \frac{2}{3} = \frac{3}{1} + \frac{3}{2} - 1 = 3\frac{1}{2}.
\end{aligned}$$

It is instructive to do this problem via conditional probabilities. *Conditioned* on the first box being an H, we are waiting for a B, an event of probability $\frac{1}{3}$ to occur. On average, an *additional* 3 boxes must be bought.. *Conditioned* on the first box being a B, we are waiting for an H, an event of probability $\frac{2}{3}$ to occur. On average, an *additional* 1.5 boxes must be bought. Hence,

$$\mathsf{E}[\mathcal{X}] = (1+3)P(H) + (1+1.5)P(B) = 4 \cdot \left(\frac{2}{3}\right) + 2.5 \cdot \left(\frac{1}{3}\right) = 3\frac{1}{2}.$$

(c) Now, Mrs Kirk buys $\mathcal{W} \geq 4$ boxes of Cornies. Consider the contents of the first two boxes.

- If the first two boxes have given Jimmy one H and one K (this has probability $\frac{4}{9}$), then in effect he has been transported back in time to the previous year since he now just has to collect one H and one K. Thus, conditioned on the first two boxes having an H and a K, $\mathcal{W} = 2 + \mathcal{X}$ where $\mathcal{X}$ was discussed in parts (a) and (b).
- If the first two boxes give Jimmy two H's (probability $\frac{4}{9}$) or two K's (probability $\frac{1}{9}$), then he has to wait for two K's (or two H's) to occur. Conditioned on this event HH (or KK), $\mathcal{W} = 2 + \mathcal{V}$ where $\mathcal{V}$ is a *negative binomial* random variable with parameters $(2, \frac{1}{3})$ (or $(2, \frac{2}{3})$) and mean 6 (or 3).

Hence, for $k \geq 4$, $P\{\mathcal{W} = k\} = P\{\mathcal{X} = k-2\}\frac{4}{9} + P\{\mathcal{V} = k-2\}\frac{4}{9} + P\{\mathcal{V} = k-2\}\frac{1}{9}$

$$= \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{k-3}\right]\frac{4}{9} + \left[(k-3)\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{k-4}\right]\frac{4}{9} + \left[(k-3)\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{k-4}\right]\frac{1}{9}$$

which simplifies to $P\{\mathcal{W} = k\} = \dfrac{(2^{k-2}+4)(k-1)}{3^k}$. Computing $\mathsf{E}[\mathcal{W}]$ from this result is messy. It is much easier to compute $\mathsf{E}[\mathcal{W}]$ via conditional probabilities. If the first two boxes give Jimmy a H and a K, his mother has to buy 3.5 more boxes on average. If he gets HH, 6 more boxes are needed on average, while if he gets KK, 3 more boxes are needed on average. Hence,

$$\mathsf{E}[\mathcal{W}] = 2 + \left(3\frac{1}{2}\right) \times \frac{4}{9} + 6 \times \frac{4}{9} + 3 \times \frac{1}{9} = 6\frac{5}{9}.$$

5. (a) Assuming that the CRC detects all errors (which is not strictly true), the probability that the CRC indicates no error in a received packet is just $(1-p)^N$, the probability that all N bits were received correctly.

(b) $P\{\text{packet lost}\} = P\{\text{packet in error 5 times}\} = (1-(1-p)^N)^5$ and hence

$$\begin{aligned}
&P\{\text{packet is received successfully}\} \\
&\qquad = 1 - (1-(1-p)^N)^5 \\
&\qquad = 5(1-p)^N - 10(1-p)^{2N} + 10(1-p)^{3N} - 5(1-p)^{4N} + (1-p)^{5N}.
\end{aligned}$$

(c) Let $Q = (1-p)^N$. Then, $P\{\mathcal{X}_i = 1\} = Q$. $P\{\mathcal{X}_i = 2\} = [1-Q]Q$. $P\{\mathcal{X}_i = 3\} = [1-Q]^2 Q$. $P\{\mathcal{X}_i = 4\} = [1-Q]^3 Q$. $P\{\mathcal{X}_i = 5\} = [1-Q]^4$. Note that the first four values are those corresponding to a geometric random variable with parameter Q, while the last is the probability that this random variable has value 5 or more..

$$\begin{aligned}
\mathsf{E}[\mathcal{X}_i] &= Q + 2[1-Q]Q + 3[1-Q]^2 Q + 4[1-Q]^3 Q + 5[1-Q]^4 \\
&= 1 + [1-Q] + [1-Q]^2 + [1-Q]^3 + [1-Q]^4 \\
&= \frac{1-[1-Q]^5}{1-[1-Q]} = 5 - 10Q + 10Q^2 - 5Q^3 + Q^4.
\end{aligned}$$

Verify that the sum telescopes from the right to the value shown.

(d) $P\{\text{none of the } L \text{ packets are lost}\} = (1-(1-(Q)^5)^L$.

6. (a) We are modeling the guesses as independent trials, and we are not allowing for other possibilities such as on some questions, the student can eliminate one or more alternatives and thus improve chances of getting the right answer to $\frac{1}{4}$ or $\frac{1}{3}$ etc.

(b) The likelihood of observation $\mathcal{W} = n$ is $\binom{N-K}{n}(0.8)^n(0.2)^{N-K-n}$ for $0 \le n \le N-K$.

(c) For given N and n, the likelihood of part (b) is a function, say $f(K)$ of K. We have that $\dfrac{f(K)}{f(K-1)} = \dfrac{\binom{N-K}{n}(0.8)^n(0.2)^{N-K-n}}{\binom{N-(K-1)}{n}(0.8)^n(0.2)^{N-(K-1)-n}} = \dfrac{N-(K-1)-n}{0.2(N-(K-1))} \ge 1$ iff $K \le N - 1.25n + 1$. Thus, the likelihood is maximum when K has value $\hat{K} = \lfloor N - 1.25n + 1 \rfloor$.

(d) If n is *not* a multiple of 4, then the analysis of part (b) shows that $f(\hat{K})$ is the unique maximum of the likelihood, and the examiner's estimate $\tilde{K} = N - n - \lfloor 0.25n \rfloor$ equals the maximum likelihood estimate. When n *is* a multiple of 4, then $f(\hat{K}) = f(\hat{K}-1)$, and thus both $\hat{K}$ and $\hat{K}-1$ are legitimate maximum likelihood estimates of the unknown quantity. Note that in this case, the examiner's estimate $\tilde{K} = \hat{K} - 1$ and thus the examiner *is* using a maximum likelihood estimate (she is just a tough grader!).

For $N = 100$ and $K = 90$, $\mathcal{W}$ can take on values $0, 1, \ldots, 10$. The value of $\mathcal{W}$ most likely to occur is 8. In this case, $\hat{K} = 91$ while $\tilde{K} = 90$ and so the examiner *does* estimate K correctly. On the other hand, if $\mathcal{W} = 4$, the examiner estimates K to be $\tilde{K} = 95$ (erring on the side of caution), and if $\mathcal{W} = 10$, then $\tilde{K} = 88$, ouch!

7. The number of red balls drawn is a binomial random variable $\mathcal{X}$ with parameters $(100, p)$ where $p = 10/(10+x)$. We are told that $\mathcal{X} = 25$ on one experiment.

(a) The maximum-likelihood estimate of x is denoted $\hat{x}$. Now, we can look at the ratio of the likelihoods of observing 25 red balls when the number of blue balls is k and $k-1$ to get

$$\frac{\binom{100}{25}(10/(10+k))^{25}(k/(10+k))^{75}}{\binom{100}{25}(10/(9+k))^{25}((k-1)/(9+k))^{75}} = \left(\frac{9+k}{10+k}\right)^{100}\left(\frac{k}{k-1}\right)^{75}$$

and try to figure out the k for which the ratio ≤ 1 which is messy. Alternatively, and more easily, we know that $\hat{p} = 0.25$ maximizes $\binom{100}{25}p^{25}(1-p)^{75}$. From this, we get that $\hat{p} = 0.25 = 10/(10+\hat{x})$, and so $\hat{x} = 30$.

(b) A confidence interval of length 0.2 gives a confidence level of $1 - 1/(100 \cdot 0.2^2) = 75\%$.

(c) A confidence level of 96% results in a confidence interval of length $1/\sqrt{100 \cdot 0.04} = 0.5$. Note that the confidence interval is $(\hat{p} - 0.25, \hat{p} + 0.25) = (0, 0.5)$.