University of Illinois Spring 2008

ECE 313: Solutions to Problem Set 4

1. (a) $P\{V=5\}=0.6, P\{V=30\}=0.4 \Rightarrow \mathsf{E}[V]=5\times 0.6+30\times 0.4=15$ miles per hour

   (b) $T=g(V)=\dfrac{120}{V}$ minutes.
   $P\{T=24\}=P\{V=5\}=0.6;\quad P\{T=4\}=P\{V=30\}=0.4$
   $\Rightarrow \mathsf{E}[T]=24\times 0.6+4\times 0.4=16$ minutes.

   (c) $g(\mathsf{E}[V])=\dfrac{120}{15}=8\neq \mathsf{E}[g(V)]=16$

2. (a) Note that: $P\{X>0\}=p_X(1)+p_X(2)+p_X(3)$; $\quad P\{X>1\}=p_X(2)+p_X(3)$; and $P\{X>2\}=p_X(3)$. Thus,

$$\begin{aligned}
\mathsf{E}[X] &= \sum_{k=1}^{3} k\cdot p_X(k)\\
&= 1\cdot p_X(1)+2\cdot p_X(2)+3\cdot p_X(3)\\
&= p_X(1)+p_X(2)+p_X(2)+p_X(3)+p_X(3)+p_X(3)\\
&= [p_X(1)+p_X(2)+p_X(3)]+[p_X(2)+p_X(3)]+[p_X(3)]\\
&= P\{X>0\}+P\{X>1\}+P\{X>2\}\\
&= \sum_{k=0}^{2} P\{X>k\}.
\end{aligned}$$

   (b) Note that $P\{X>k\}=p_X(k+1)+p_X(k+2)+p_X(k+3)+\cdots$. Thus,

$$\begin{array}{llllllllll}
\displaystyle\sum_{k=0}^{\infty} P\{X>k\} = P\{X>0\} & + & P\{X>1\} & + & P\{X>2\} & + & P\{X>3\} & +\cdots \\
\quad = p_X(1) & + & p_X(2) & + & p_X(3) & + & p_X(4) & +\cdots \\
 & + & p_X(2) & + & p_X(3) & + & p_X(4) & +\cdots \\
 & & & + & p_X(3) & + & p_X(4) & +\cdots \\
 & & & & & + & p_X(4) & +\cdots \\
 & & & & & & & +\cdots \\
 & & & & & & & \ddots \\
\quad = 1\cdot p_X(1) & + & 2\cdot p_X(2) & + & 3\cdot p_X(3) & + & 4\cdot p_X(4) & +\cdots
\end{array}$$

$$= \sum_{k=1}^{\infty} k\cdot p_X(k) = \sum_{k=0}^{\infty} k\cdot p_X(k) = \mathsf{E}[X].$$

   (c) For a geometric random variable with parameter p,
   $P\{X>k\}=(1-p)^k p+(1-p)^{k+1}p+(1-p)^{k+2}p+\cdots=(1-p)^k p\displaystyle\sum_{i=0}^{\infty}(1-p)^i$
   $=(1-p)^k p\times\dfrac{1}{1-(1-p)}=(1-p)^k$. More directly, the event $\{X>k\}$ occurs if and only if the first k trials ended in failure, and this has probability $(1-p)^k$.
   $\mathsf{E}[X]=\displaystyle\sum_{k=0}^{\infty}P\{X>k\}=\sum_{k=0}^{\infty}(1-p)^k=\frac{1}{1-(1-p)}=\frac{1}{p}$ as before.

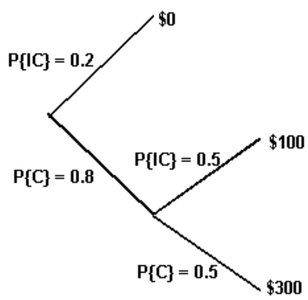
   (d) $\mathsf{E}[X]=\displaystyle\sum_{k=1}^{\infty}k\cdot p_X(k)=\sum_{k=1}^{\infty}k(1-p)^{k-1}p=p\sum_{k=0}^{\infty}(k+1)(1-p)^k=p\cdot\frac{1}{(1-(1-p))^2}=\frac{1}{p}.$

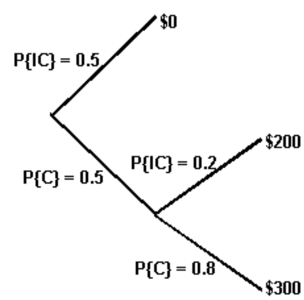
3. (a) Your net worth decreases by \$1 with probability $\frac{31}{32}$ and increases by \$28 with probability $\frac{1}{32}$ for an average *loss* of $\$\frac{3}{32}$ per drawing. A fair bet would have a prize of \$32, not \$29.
   (b) Let X denote the number of the trial on which 7 occurs for the first time. Then, X is a geometric random variable with parameter $\frac{1}{32}$. Your winnings are

$$\$Y = \begin{cases} \$29 - X, & X \leq 29, \\ -\$29, & X > 29, \end{cases}$$

   and LOTUS gives $\$\mathsf{E}[\mathrm{Y}] = -\$1.80529\ldots$. On average, you lose less than $(29 \times \$\frac{3}{32})$, because you don't always bet on all 29 draws; you prudently stop when you have won.
   (c) $P\{X > 21\} = \left(1 - \frac{1}{32}\right)^{21} = 0.5133884$, and so you lose \$1 with probability 0.5133884 and win \$1 with probability $1 - 0.5133884 = 0.4866116$ for an average loss of \$ 0.0267767 per Minibucks ticket.

4. (a) $p_X(100) = 0.8, p_X(0) = 0.2 \Rightarrow \mathsf{E}[X] = 100 \times 0.8 + 0 \times 0.2 = \$80$
   (b) $p_X(200) = 0.5, p_X(0) = 0.5 \Rightarrow \mathsf{E}[X] = 200 \times 0.5 + 0 \times 0.5 = \$100$
   (c) To maximize $\mathsf{E}[\mathrm{X}]$, choose Question 2.





   (d) (See figure above left) $p_Y(0) = 0.2, p_Y(100) = 0.8 \times 0.5 = 0.4, p_Y(300) = 0.8 \times 0.5 = 0.4$ $\Rightarrow \mathsf{E}[Y] = 100 \times 0.4 + 300 \times 0.4 = \$160$.
   (e) (See figure above right) $p_Y(0) = 0.5, p_Y(200) = 0.5 \times 0.2 = 0.1, p_Y(300) = 0.5 \times 0.8 = 0.4$ $\Rightarrow \mathsf{E}[Y] = 200 \times 0.1 + 300 \times 0.4 = \$140$.
   (f) To maximize $\mathsf{E}[\mathrm{Y}]$, choose Question 1 first.
   (g) For each question, we know the probability that we will answer it correctly and the prize money we get if we answer it correctly. In the first game, we just compare the expected earnings for answering each question and we pick Question 2 since it maximizes the expected earnings. In the second game, getting a chance to answer both Questions has a better payoff than answering just one Question. So, we choose the easier Question (Question 1 in this instance) as the one to answer first. (Actually, this argument is not completely correct because as part (h) below shows, if the prize for the easier Question is very small, it is best to just take one's chances on the harder Question first.)
   (h) If we choose Question 1 first $\Rightarrow \mathsf{E}[Y_1] = p_1 v_1 + p_1(p_2 v_2)$
   If we choose Question 2 first $\Rightarrow \mathsf{E}[Y_2] = p_2 v_2 + p_2(p_1 v_1)$
   Choose Question 1 first iff

$$\begin{aligned} \mathsf{E}[Y_1] &\geq \mathsf{E}[Y_2] \\ p_1 v_1 + p_1(p_2 v_2) &\geq p_2 v_2 + p_2(p_1 v_1) \\ \frac{p_1 v_1}{1 - p_1} &\geq \frac{p_2 v_2}{1 - p_2} \end{aligned}$$

   If we use the probabilities given in the problem, we see that the easier Question 1 should be answered first *unless* the prizes are such that $v_1 < 0.25 v_2$ in which case the harder Question 2 should be answered first.

5. Each die has probability $\frac{1}{6}$ of matching your chosen number. Thus, the number of dice that match is a binomial random variable with parameters $(3, \frac{1}{6})$.

   (a) You lose \$6 if none of the dice show your chosen number. This occurs with probability $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$. If one of the dice matches, you win \$6 with probability $3 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 = \frac{75}{216}$. You win \$12 with probability $3 \cdot \frac{5}{6} \cdot \left(\frac{1}{6}\right)^2 = \frac{15}{216}$, and you win \$18 with probability $\left(\frac{1}{6}\right)^3 = \frac{1}{216}$. Rationality test: $125 + 75 + 15 + 1 = 216$.

   (b) See results from Part (a).

   (c) $\mathsf{E}[X] = (-6)\dfrac{125}{216} + 6\dfrac{75}{216} + 12\dfrac{15}{216} + 18\dfrac{1}{216} = -\dfrac{102}{216}$.

   (d) If all three dice show different numbers (which has probability $\frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{120}{216}$, you win \$1 on the three numbers but lose \$1 on the other three for a net gain of \$0. If two dice show the same number (which has probability $3 \cdot \frac{6 \cdot 1 \cdot 5}{6 \cdot 6 \cdot 6} = \frac{90}{216}$ of occurring) you win \$2 for that number and \$1 for the other number showing, but lose \$1 on the remaining four for a net loss of \$1. If all three dice show the same number (which has probability $6 \cdot \frac{1 \cdot 1 \cdot 1}{6 \cdot 6 \cdot 6} = \frac{6}{216}$ of occurring, you win \$3 for that number, but lose \$1 on the other five for a net loss of \$2. Thus,

$$\mathsf{E}[Y] = (0)\frac{120}{216} + (-1)\frac{90}{216} + (-2)\frac{6}{216} = -\frac{102}{216}.$$

   Thus, the average loss is the same regardless of whether you split your bet or not. Worse yet, your wealth will never increase if you split your bet: you will *never* win any money, whereas if you bet your money on one number, there is a small chance that you will be ahead at some time. (Do remember to quit while you are ahead, will ya?)

6. (a) The Taylor series expansion of function e^λ at $x = 0$ (or MacLaurin series) is

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots + \frac{\lambda^k}{k!} + \cdots.$$

   Therefore, the sum of the pmf $p_X(k) = \displaystyle\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}e^\lambda = 1$.

   (b)

$$\mathsf{E}[X] = \sum_{k=0}^{\infty} e^{-\lambda} k \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda}\lambda\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = \lambda e^{-\lambda}e^\lambda = \lambda.$$

   (c) Again by LOTUS,

$$\mathsf{E}[X(X-1)] = \sum_{k=0}^{\infty} e^{-\lambda}k(k-1)\frac{\lambda^k}{k!} = \sum_{k=2}^{\infty} e^{-\lambda}\lambda^2\frac{\lambda^{k-2}}{(k-2)!} = \lambda^2 e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = \lambda^2.$$

   (d) $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = \mathsf{E}[X^2 - X] + \mathsf{E}[X] - (\mathsf{E}[X])^2 = \lambda$.
   Note that when you need to find $\mathsf{E}[X^2]$ and the pmf has $k!$ in the denominator, it is a convenient trick to compute $\mathsf{E}[X(X-1)]$ and then find $\mathsf{E}[X^2] = \mathsf{E}[X(X-1)] + \mathsf{E}[X]$. This because we can cancel the $k(k-1)$ from the numerator and denominator. The trick also works, for example, in finding the variance of a binomial random variable.

   (e) By LOTUS, $\mathsf{E}[z^X] = \displaystyle\sum_{k=0}^{\infty} z^k e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda}\frac{(\lambda z)^k}{k!} = e^{-\lambda}e^{z\lambda} = e^{\lambda(z-1)}$.