ECE 313: Solutions to Problem Set 3

1. (a) $\Gamma(1) = \int_0^\infty \exp(-t)\,dt = -\exp(-t)\Big|_0^\infty = 1.$

   (b) For $\alpha > 0$, $\Gamma(\alpha+1) = \int_0^\infty t^\alpha \exp(-t)\,dt = t^\alpha[-\exp(-t)]\Big|_0^\infty - \int_0^\infty \alpha t^{\alpha-1}[-\exp(-t)]\,dx$
   $= 0 + \alpha \int_0^\infty t^{\alpha-1}\exp(-t)\,dt = \alpha\Gamma(\alpha).$
   Hence $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2\cdot 1\cdot\Gamma(1) = n!$.

   (c) Similarly, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha(\alpha-1)\Gamma(\alpha-1) = \cdots = \alpha(\alpha-1)(\alpha-2)\cdots\Gamma(\alpha-\lfloor\alpha\rfloor)$ where $\lfloor\alpha\rfloor$ is the *integer part* of α, and $0 < \alpha - \lfloor\alpha\rfloor < 1$.

   (d) Using the substitution $x = \sqrt{2t}$, $dt/\sqrt{t} = \sqrt{2}dx$ and the suggested change to polar coordinates,

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2}\exp(-t)\,dt = \sqrt{2}\int_0^\infty \exp(-x^2/2)\,dx = \sqrt{2}\int_0^\infty \exp(-y^2/2)\,dy \\
&= \left(2\int_0^\infty\int_0^\infty \exp(-[x^2+y^2]/2)\,dx\,dy\right)^{1/2} = \left(2\int_{r=0}^\infty\int_{\theta=0}^{\pi/2} \exp(-r^2/2)\cdot r\,d\theta\,dr\right)^{1/2} \\
&= \left(\pi\int_{r=0}^\infty r\exp(-r^2/2)dr\right)^{1/2} = \sqrt{\pi}\ \text{ via the result of Problem 5(b) of Problem Set 1.}
\end{aligned}
$$

2. (a) $\binom{m+n}{k}$ committees of size k can be drawn from a set of m men and n women. Of these, $\binom{m}{i}\binom{n}{k-i}$ committees have i men and $k-i$ women. Summing over all possible values of i, we conclude that $\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$.

   (b) The coefficient of x^k in the polynomial $(1+x)^{m+n}$ is $\binom{m+n}{k}$. Now, if $h(x) = f(x)g(x)$, then the coefficient h_k of $h(x)$ is the discrete convolution of the coefficients of $f(x)$ and $g(x)$: *viz.*, $h_k = \sum_i f_i g_{k-i}$. Applying this to $f(x) = (1+x)^m$ and $g(x) = (1+x)^n$, we get that $\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$ as before.

   (c) With $m = k = n$, $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n}\left[\binom{n}{i}\right]^2$ since $\binom{n}{n-i} = \binom{n}{i}$.

3. Consider the experiment of picking a person at random from the 100,000 population of the town, and let A, B, and C denote the events that the person selected reads Newspapers I, II, and III respectively. We are given that $P(A) = 0.1, P(B) = 0.3, P(C) = 0.05$; $P(AB) = 0.08, P(AC) = 0.02, P(BC) = 0.04$; and $P(ABC) = 0.01$. Marking these probabilities on a Karnaugh map, we readily deduce the probabilities shown in the figure on the next page. (Venn diagramists and those who believe in brainpower alone unaided by any diagrams can work it out that $P(AB) = P(ABC) + P(ABC^c)$ can be used to deduce that $P(ABC^c) = 0.07$ etc.) From the diagram, it is easy to read off that

   (a) $P\{\text{read one newspaper only}\} = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC) = 0.2 \Rightarrow 20,000$ people read only one newspaper.

   (b) $P\{\text{read at least two newspapers}\} = P(AB \cup AC \cup BC) = 0.12 \Rightarrow 12,000$ people read at least two newspaper.

   (c) $P\{\text{read at least one of I and III and also read II}\} = P((A\cup C)\cap B) = 0.11 \Rightarrow 11,000$ people read at least one of I and III and also read II.

(d) $P\{\text{read no newspapers}\} = P(A^cB^cC^c) = 0.68 \Rightarrow 68,000$ people read no newspapers.

(e) $P\{\text{read exactly one of I and III and also read II}\} = P((A \oplus C) \cap B) = 0.1 \Rightarrow 10,000$ people read exactly one of I and III and also read II.

	B^c	B	B	B^c
C^c	0.68	0.19	0.07	0.01
C	0.00	0.03	0.01	0.01
	A^c	A^c	A	A

4. $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem. Thus,

(a) $B \cap C = \emptyset$ and hence $P(A \cup (B \cap C)) = P(A \cup \emptyset) = P(A) = 1/3$.

(b) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C))$
$= 4P(A \cap B \cap C) + 2P(A \cap B \cap C) - P(A \cap B \cap C) = 5P(A \cap B \cap C) = 5/8$.

(c) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = 1/2 + 1/3 - 0 = 5/6$. Why?

(d) $(A \cup (B^c \cup C^c)^c)^c = A^c \cap (B^c \cup C^c)$. Hence, $P(A \cup (B^c \cup C^c)^c) = 1 - 0.6 = 0.4$.

5. (a) i. The sample space consists of 5-tuples of the form (B, C, B, B, C) where B and C have the obvious meaning, and 3 of the entrees must be B and 2 must be C. Obviously, the broccolous days can be chosen in $\binom{5}{3} = 10$ ways and hence $|\Omega| = 10$.

ii. $P(\text{broccoli on Monday}) = \binom{4}{2}/\binom{5}{2} = 3/5$.

iii. $P(\text{broccoli on Monday and Friday}) = \binom{3}{2}/\binom{5}{2} = 3/10$.

iv. $P(\text{broccoli on Monday, Wednesday, and Friday}) = 1/\binom{5}{2} = 1/10$.

(b) We now have 3 independent trials of an experiment.

i. $P(\text{same veg on three days}) = P(AAA) + P(BBB) + P(CCC) = (0.2)^3 + (0.5)^3 + (0.3)^3 = 0.16$.

ii. Let $A^c = B \cup C$. Then, $P(\text{same veg on two days}) = 3[P(AAA^c) + P(BBB^c) + P(CCC^c)]$
$= 3[(0.2)^2 \times 0.8 + (0.5)^2 \times 0.5 + (0.3^2) \times 0.7] = 0.66$. Why the factor 3?

iii. $P(\text{three different}) = 3!P(A)P(B)P(C) = 0.18$. But why 3!?? Alternatively, we can compute this as $1 - 0.16 - 0.66 = 0.18$. Why?

6. (a) i. $P(\Omega) = \displaystyle\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{2(n!)} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} = \frac{1}{2}\exp(\ln 2) = 1$.

ii. $P(n \text{ is even}) = \displaystyle\sum_{n=0}^{\infty} \frac{(\ln 2)^{2n}}{2(2n)!}$. Now, $\exp(x) + \exp(-x) = 2\cosh(x) = 2\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$. Hence, $P(n \text{ is even}) = (1/4)[\exp(\ln 2) + \exp(-\ln 2)] = 5/8$.

(b) $P(B) = P(H) + P(TTTTH) + P(TTTTTTTTH) + \cdots = p + q^4p + q^8p + \cdots = \dfrac{p}{1-q^4}$.

$P(C) = P(TH) + P(TTTTTH) + P(TTTTTTTTTH) + \cdots = q[p + q^4p + q^8p + \cdots] = \dfrac{pq}{1-q^4}$.

Similarly, we have that $P(T) = \dfrac{pq^2}{1-q^4}$ and $P(A) = \dfrac{pq^3}{1-q^4}$.

A nicer way of expressing these results is to note that $1 - q^4 = (1-q)(1+q+q^2+q^3)$ and therefore $P(B) = \dfrac{1}{1+q+q^2+q^3}$, $P(C) = \dfrac{q}{1+q+q^2+q^3}$, $P(T) = \dfrac{q^2}{1+q+q^2+q^3}$, $P(A) = \dfrac{q^3}{1+q+q^2+q^3}$. Since $q < 1$, we have that $P(B) > P(C) > P(T) > P(A)$ which perhaps explains why Alice doesn't live here anymore. Also, quite obviously, $P(B) + P(C) + P(T) + P(A) = 1$.