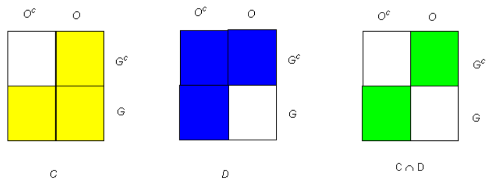
University of Illinois Spring 2008

ECE 413: Solutions to Problem Set 2

1. Define events $C = \{$At least one of O and G occurs$\}$ and $D = \{$At least one of O and G does not occur$\}$. The Karnaugh map below shows the various events of interest:



We are given that $P(C) = P(O \cup G) = 0.65$ and $P(D) = P(O^c \cup G^c) = 0.7$. Now, note that $C \cup D = \Omega$ while $C \cap D = (O \cup G) \cap (O^c \cup G^c) = (O \cap G^c) \cup (O^c \cap G) = O \oplus G$. Consequently, $P(C \cup D) = P(\Omega) = 1 = P(C) + P(D) - P(O \oplus G) = 0.65 + 0.7 - P(O \oplus G)$ from which we get that $P(O \oplus G) = 0.35$.

2.
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \cdots$$
$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \binom{n}{4}x^4 - \cdots$$

from which we get that $(1+x)^n + (1-x)^n = 2\left[\binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots\right]$. Set $x = 1$ and note that the left side has value 2^n while the right side is twice the number of sets with an even number of elements. More simply, set $x = 1$ in $(1-x)^n$ to get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \ \Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$$

3. (a) For any event A, $P(A) \leq 1$. Hence, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$. It follows that
$$P(A \cap B) \geq P(A) + P(B) - 1$$

(b) i. Since A is a subset of $A \cup B$, we know that $P(A) \leq P(A \cup B)$. Similarly $P(B) \leq P(A \cup B)$. Adding the two inequalities gives that $P(A) + P(B) \leq 2P(A \cup B)$, and thus it follows that
$$\frac{P(A) + P(B)}{2} \leq P(A \cup B)$$
with equality iff $P(A) = P(B) = P(A \cup B)$ or equivalently, $P(A) = P(B) = P(A \cap B)$
To prove the other inequality, note that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A and B. Since $P(A \cap B) \geq 0$, it follows that
$$P(A \cup B) \leq P(A) + P(B)$$
with equality iff $P(A \cap B) = 0$. This bound is sometimes referred to as the *union bound*.

ii. As shown in the previous part, $P(A) \leq P(A \cup B \cup C)$, $P(B) \leq P(A \cup B \cup C)$ and $P(C) \leq P(A \cup B \cup C)$. Adding the three inequalities gives that $P(A) + P(B) + P(C) \leq 3P(A \cup B \cup C)$, and thus it follows that
$$\frac{P(A) + P(B) + P(C)}{3} \leq P(A \cup B \cup C)$$
with equality iff $P(A) = P(B) = P(C) = P(A \cup B \cup C)$.
In the previous part, we proved that $P(A \cup B) \leq P(A) + P(B)$ holds for events A and B. Let D denote $A \cup B$. Then,
$$P(A \cup B \cup C) = P(D \cup C) \leq P(D) + P(C) = P(A \cup B) + P(C) \leq P(A) + P(B) + P(C).$$
Equality holds iff $P(A \cap B) = P(A \cap C) = P(B \cap C) = 0$ (which implies that $P(A \cap B \cap C) = 0$.)

4. (a) Each club must have at least one member (*i.e.* the leader) who can be chosen to be any of the n FOMDLIUans. The remaining members of the club can be *any* subset of the remaining $n-1$ FOMDLIUans. Since there are 2^{n-1} such subsets, we conclude that the number of clubs is $n2^{n-1}$. More laboriously, we have $n = n\binom{n-1}{0}$ possibilities for clubs with 1 member, $n\binom{n-1}{1}$ possibilities for clubs with 2 members, $n\binom{n-1}{2}$ possibilities for clubs with three members, and so on. Putting this together, we get:

$$\begin{aligned}
\text{Number of clubs} &= n\binom{n-1}{0} + n\binom{n-1}{1} + n\binom{n-1}{2} + \cdots + n\binom{n-1}{n-2} + n\binom{n-1}{n-1} \\
&= n\left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-2} + \binom{n-1}{n-1}\right] \\
&= n2^{n-1}.
\end{aligned}$$

(b) Using the previous part, we see that the number of clubs with exactly k members is given by $n\binom{n-1}{k-1}$ which is the same as $k\binom{n}{k}$, as was proved in class. The first displayed sum in 4(a) is thus $\sum_{k=1}^{n} k\binom{n}{k}$ which we counted to be $n2^{n-1}$ in part (a). More explicitly, we count the number of clubs by first selecting the members and then choosing the leader from among the members. Fix a number of members k for a particular club. From a pool of n individuals, there are $\binom{n}{k}$ possibilities. Once we select a club of size k, there are now k possibilities to select a leader. Therefore the total number of clubs is given by:

$$\binom{n}{1}\cdot 1 + \binom{n}{2}\cdot 2 + \cdots + \binom{n}{n}\cdot n = \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

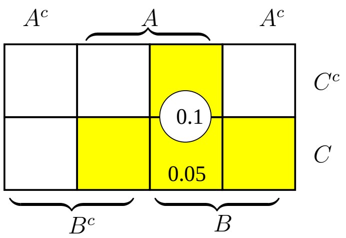
(c)

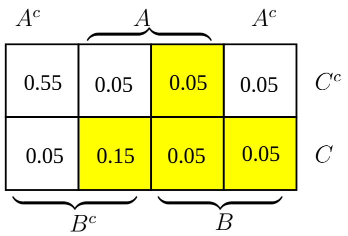
$$\begin{aligned}
\frac{d}{dx}\left[(1+x)^n\right] &= n(1+x)^{n-1} \quad \text{(by the Chain Rule)} \\
(1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \quad \text{(Binomial Theorem)} \\
\frac{d}{dx}\left[(1+x)^n\right] &= \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + 4\binom{n}{4}x^3 + \cdots
\end{aligned}$$

Evaluate the derivative of $(1+x)^n$ at $x=1$ in two different ways and equate the results to get

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}.$$

5. (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event $(A\cap B)\cup(B\cap C)\cup(A\cap C)$.

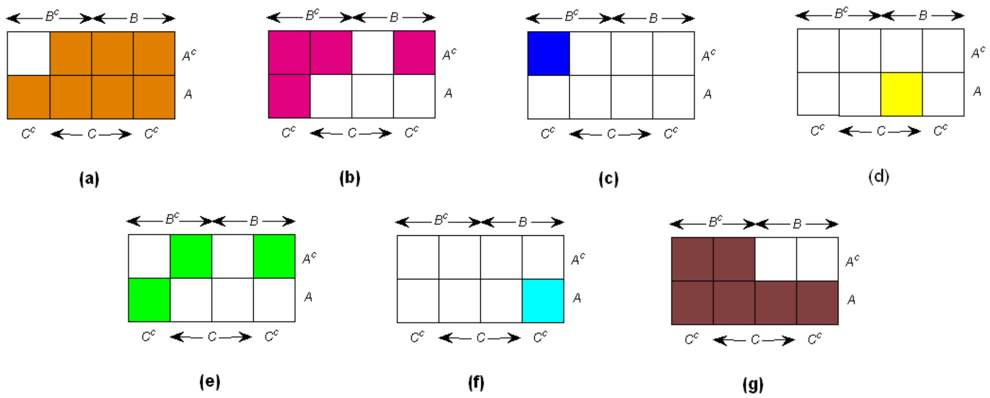




(b) Since $A\cap B$ is the disjoint union of $A\cap B\cap C$ and $A\cap B\cap C^c$, we get that $P(A\cap B) = 0.1 = P(A\cap B\cap C) + P(A\cap B\cap C^c) = 0.05 + P(A\cap B\cap C^c)$ giving that $P(A\cap B\cap C^c) = 0.05$. Since $P(AB\cup BC\cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$ while $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$, we readily obtain that $P(AB^cC) = 0.15$ and $P(A^cBC) = 0.05$. Since $P(A\cup B) = P(A) + P(B) - P(A\cap B) = 0.6$ we have that $P(A^cB^c) = 1 - P(A\cup B) = 0.4$. Since $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$, we get that $P(\text{cereal snaps, crackles, and pops}) = P(A^c\cap B^c\cap C^c) = 0.55$.

(c) $P(\text{the sample fails } only \text{ the snap test}) = P(AB^cC^c) = 0.05$.
$P(\text{the sample fails } only \text{ the crackle test}) = P(A^cBC^c) = 0.05$.
$P(\text{the sample fails } only \text{ the pop test}) = P(A^cB^cC) = 0.05$.

6. (a) At least one of the events A, B, C occurs; – $A \cup B \cup C = (A^cB^cC^c)^c$
   (b) At most one of the events A, B, C occurs; – $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC) \cup (A^cB^cC^c) = (AB \cup BC \cup AC)^c$
   (c) None of the events A, B, C occurs; – $(A \cup B \cup C)^c = A^cB^cC^c$ [compare to part (a)]
   (d) All three events A, B, C occur; – ABC
   (e) Exactly one of the events A, B, C occurs; – $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC)$
   (f) Events A and B occur, but not C; – ABC^c
   (g) Either event A occurs, or if not then B also does not occur; – $A \cup A^cB^c = A \cup B^c$



7. (a) There are $\binom{5}{2} = 10$ games in this tournament, and $\binom{n}{2} = \dfrac{n(n-1)}{2}$ games in general.
   (b) Yes. In fact, many schedules can be specified. Here is one: imagine that the teams have been arranged in a circle so that each team can be thought of as having two teams on its left and two teams on its right. Then, each team wears home uniforms when playing a team on its left and away uniforms when playing a team on its right.
   (c) There are $2^{10} = 1024$ possible outcomes of this tournament.
      i. Two teams cannot possibly have 4-0 records. However, if Team A, say, has a 4-0 record, then we know what happened in 4 games, while the outcomes of the remaining 6 games are arbitrary. Hence, $P(\text{Team A has a 4-0 record}) = \dfrac{2^6}{2^{10}} = \dfrac{1}{2^4} = \dfrac{1}{16}$ and $P(\textit{some} \text{ team has a 4-0 record}) = 5 \times \dfrac{1}{16} = \dfrac{5}{16}$. (What axiom are we using here?)
      ii. The same argument shows that $P(\textit{some} \text{ team has a 0-4 record}) = \dfrac{5}{16}$.
      iii. If Team A wins all four of its games *and* Team B loses all four of its games, then we know what happened in 7 games (why not 8?), and hence we get that $P(\text{Team A has 4-0 record; Team B has a 0-4 record}) = \dfrac{2^3}{2^{10}} = \dfrac{1}{2^7} = \dfrac{1}{128}$, and $P(\text{some team has 4-0 record; some other team has a 0-4 record}) = 20 \times \dfrac{1}{128} = \dfrac{5}{32}$.
      iv. The remaining teams have lost one game (against the 4-0 team) and won another game (against the 0-4 team), and will have identical 2-2 records if each wins one game and loses one game among the three games that these teams play against one another (e.g. A beats B who beats C who beats A: basketball is not necessarily a transitive game!). Since only 2 of the 8 outcomes of these three games give 2-2 records for all three teams, we get that $P(\text{one team is 4-0; another is 0-4; rest are 2-2}) = \dfrac{5}{32} \times \dfrac{2}{8} = \dfrac{5}{128}$.