ECE 313: Solutions to Problem Set 1

1. (a) Let $n > 0$ be an integer. Then,
$$(1-x)\left[1+x+x^2+\cdots+x^{n-1}\right] = \left[1+x+x^2+\cdots+x^{n-1}\right] - \left[x+x^2+\cdots+x^n\right] = 1-x^n.$$
If $x \neq 1$, divide both sides by $(1-x)$ to get $1+x+x^2+\cdots+x^{n-1} = \dfrac{1-x^n}{1-x}$.

(b) Since $|x| < 1$, $1+x+x^2+\cdots = \lim_{n\to\infty} 1+x+x^2+\cdots+x^{n-1} = \lim_{n\to\infty} \dfrac{1-x^n}{1-x} = \dfrac{1}{1-x}$.

(c) The 0-th derivative $f^{(0)}(x)$ is just $f(x) = (1+x)^n$ itself. The first derivative is $f^{(1)}(x) = n(1+x)^{n-1}$, the second is $f^{(2)}(x) = n(n-1)(1+x)^{n-2}$, and so on. The k-th derivative is $f^{(k)}(x) = n(n-1)\cdots(n-k+1)(1+x)^{n-k}$, and finally the n-th derivative is $f^{(n)}(x) = n(n-1)\cdots 2\cdot 1 = n!$ which is a constant. Hence, the first $n+1$ terms of the Maclaurin series for $f(x)$ are
$$f(x) = (1+x)^n \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}x^k.$$

(d) Since $f^{(n)}(x)$ is a constant, $f^{(k)}(x) = 0$ for all $k > n$. Thus, the above *is* the complete Maclaurin series for $f(x)$; there is *no* approximation.

(e) $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2\cdot 1}{((n-k)(n-k-1)\cdots 2\cdot 1)k!} = \dfrac{n(n-1)\cdots(n-k+1)}{k!}$ and so the results match.

(f) Using the chain rule,
$$\begin{aligned} g^{(k)}(x) &= (-n)(-n-1)(-n-2)\cdots(-n-k+1)(1-x)^{-n-k}(-1)^k \\ &= n(n+1)(n+2)\cdots(n+k-1)(1-x)^{-n-k} \end{aligned}$$
Thus, $g^{(k)}(0) = n(n+1)(n+2)\cdots(n+k-1) \neq 0$ for all integers $k \geq 0$. Hence, the MacLaurin series for $g(x)$ contains terms of all degrees, and we have that
$$g(x) = (1-x)^{-n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{n(n+1)\cdots(n+k-1)}{k!}x^k.$$

(g) For $n = 1$, $n(n+1)\cdots(n+k-1) = k!$ while for $n = 2$, $n(n+1)\cdots(n+k-1) = (k+1)!$. Hence,
$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1+x+x^2+\cdots \quad \text{and} \quad (1-x)^{-2} = \sum_{k=0}^{\infty}(k+1)x^k = 1+2x+3x^2+\cdots.$$

(h) The k-th derivative of $h(x) = (1+x)^a$ where a is a real number and not necessarily an integer is $h^{(k)}(x) = a(a-1)\cdots(a-k+1)(1+x)^{a-k}$ where the exponent cannot equal 0 when a is not an integer. Thus, the MacLaurin series has terms of all degrees and we have that
$$h(x) = (1+x)^a = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!}x^k.$$

2. (a) Since $\sin x = x - x^3/3! + x^5/5! - \cdots \approx x - x^3/6$ for small x, we have
$$\frac{1}{\sin^2 x} - \frac{1}{x^2} \approx \frac{1}{x^2}\left[\frac{1}{(1-x^2/6)^2} - 1\right] = \frac{1}{x^2}\left[1 + 2\left(\frac{x^2}{6}\right) + 3\left(\frac{x^2}{6}\right)^2 + \cdots - 1\right] = \frac{1}{3} + \frac{x^2}{12} + \cdots$$
and so the limiting value as $x \to 0$ is $1/3$.

(b) $x^n \exp(-ax)$ has maximum value $\left(\dfrac{n}{a}\right)^n \exp(-n)$ at $x = n/a$. Here, $n = 25$ and $a = \ln(1.00001)$ giving a maximum value of $0.123365\ldots \times 10^{150}$ at $x = 2500012.5\ldots$.

3. (a) $\int_{-1}^{2} |x|dx = \int_{-1}^{0} -x\ dx + \int_{0}^{2} x\ dx = \left.\frac{-x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{2} = \frac{1}{2} + 2 = 2.5.$

The substitution $y = 1 - x$ gives

$$\int_{-2}^{1} x(1-x)^{19}dx = \int_{3}^{0} (1-y)y^{19}(-dy) = \int_{0}^{3} y^{19} - y^{20}\ dy = \frac{3^{20}}{20} - \frac{3^{21}}{21} = -\frac{13 \times 3^{20}}{140}.$$

(b) *No* nonnegative function $f(x)$ can satisfy $\int_{-2}^{1} f(x)\ dx < 0$. The comparison test says that if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x\ dx \geq \int_a^b g(x)\ dx$. Now let $g(x) = 0$. More simply, if the curve $f(x)$ lies above the x axis in an interval $[a, b]$, the area under the curve between a and b cannot be negative.

(c) i. False: the chain rule gives $\frac{d}{dx} f(-x) = -g(-x)$.

ii. False: the chain rule gives $\frac{d}{dx} f(x^2) = g(x^2) \cdot 2x$.

iii. True: according to the chain rule.

iv. False: the chain rule gives $\frac{d}{dx} \exp(f(x^2)) = \exp(f(x^2))g(x^2) \cdot 2x$.

v. False: the antiderivative of $g(-x)$ is $-f(-x)$, (cf i. above.)

vi. False: the antiderivative of $g(x^2)$ need not be related to $f(x^2)$ at all.

(d) i. $I = \int_{-1}^{1} \frac{2}{1+x^2} dx = 2 \cdot \arctan(x)\Big|_{-1}^{1} = 2 \cdot \left(\frac{\pi}{4} - \frac{-\pi}{4}\right) = \pi.$

ii. The substitution $y = 1/x$ changes the integrand to $\frac{2}{1+1/y^2}(-1/y^2)dy = \frac{-2}{1+y^2}\ dy$ which is the integrand for J. However, as x varies from -1 to 0, y varies from -1 to $-\infty$. Similarly, as x varies from 0 to 1, y varies from ∞ to 1. Therefore, we get

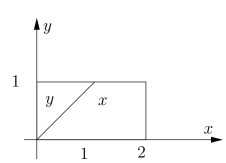
$$\int_{-1}^{1} \frac{2}{1+x^2} dx = \int_{-1}^{-\infty} \frac{-2}{1+y^2}\ dy + \int_{\infty}^{1} \frac{-2}{1+y^2}\ dy = \int_{-\infty}^{-1} \frac{2}{1+y^2}\ dy + \int_{1}^{\infty} \frac{2}{1+y^2}\ dy = \pi.$$

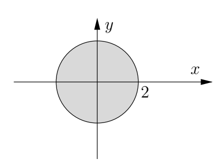
Thus, the substitution $y = 1/x$ *does not* change I into J as asserted in the problem statement.

iii. No need to re-write the math texts: $\pi \neq 0$.

4. (a) $f(x, y) = \max(x, y)$ takes on values x and y in the regions indicated in the figure below. Hence,

$$\int_{y=0}^{1}\int_{x=0}^{2} f(x,y)\ dx\ dy = \int_{y=0}^{1}\int_{x=0}^{y} y\ dx\ dy + \int_{y=0}^{1}\int_{x=y}^{2} x\ dx\ dy = \int_{y=0}^{1} y^2 + 2 - y^2/2\ dy = \frac{13}{6}.$$





(b) The integral is over the exterior of a circle of radius 2. Thus, a change to polar coordinates gives

$$\iint_{x^2+y^2>4} (x^2+y^2)^{-2}\ dx\ dy = \int_{r=2}^{\infty}\int_{\theta=0}^{2\pi} \frac{1}{r^4}\ r\ d\theta\ dr = 2\pi \left.\frac{-1}{2r^2}\right|_{r=2}^{\infty} = \frac{\pi}{4}.$$

5. (a) $\frac{d}{dx} \exp\left(-\frac{x^2}{2}\right) = -x \exp\left(-\frac{x^2}{2}\right).$

(b) From part (a), $\int_0^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx = -\exp\left(-\frac{x^2}{2}\right)\Big|_0^{\infty} = 1.$

(c) The integrand is an odd function and hence the integral has value 0.