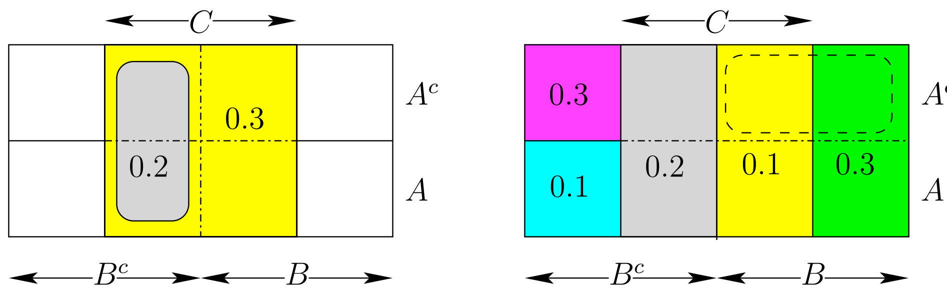
University of Illinois Spring 2008

ECE 313: Solutions to Hour Exam I

1. [30 points] Let A, B, and C denote three events defined on a sample space Ω, and suppose that $P(A) = 0.5, P(B) = 0.4, P(C) = 0.3, P(B^c \cap C) = 0.2$, and $P(A \cap B^c \cap C^c) = 0.1$.

   Find the following probabilities: $P(B \cap C)$, $P(B \cap C^c)$, $P((A \cup B \cup C)^c)$, $P(A^c \cap B)$, and $P(B^c \cap C^c)$.

   If any probability cannot be computed from the given data, check the corresponding box and leave the answer area blank.



   Solution: Karnaugh maps such as the ones shown above are very helpful in solving problems such as this. All the probabilities marked on the figures are very easily worked out almost by inspection. We have that $P(B \cap C) = 0.1, P(B \cap C^c) = P((A \cup B \cup C)^c) = 0.3$, and $P(B^c \cap C^c) = 0.4$ while $P(A^c \cap B)$, the probability of the set indicated by a dotted line on the right-hand figure above, cannot be computed. Notice that the value of $P(A)$ (which must lie between 0.1 and 0.7 (why?)) is not used except in deciding that $P(A^c \cap B)$ cannot be computed: if $P(A) = 0.1$, then $P(A^c \cap B) = 0.4$; if $P(A) = 0.7$, then $P(A^c \cap B) = 0$. For intermediate values of $P(A)$, we can't compute $P(A^c \cap B)$. Do you see why?

   For the doofi who mulishly refuse to draw Karnaugh maps, here is a solution via set algebra.
   Since C is the disjoint union of $B \cap C$ and $B^c \cap C$, we have that $P(C) = 0.3 = P(B \cap C) + P(B^c \cap C)$ $= P(B \cap C) + 0.2 \Rightarrow P(B \cap C) = 0.1$.
   Since B is the disjoint union of $B \cap C$ and $B \cap C^c$, we have that $P(B) = 0.4 = P(B \cap C) + P(B \cap C^c)$ $= 0.1 + P(B \cap C^c) \Rightarrow P(B \cap C^c) = 0.3$.
   Since $A \cup B \cup C = (B \cup C) \cup A$ is the disjoint union of $B \cup C$ and $(B \cup C)^c \cap A = A \cap B^c \cap C^c$, we have that $P(A \cup B \cup C) = P(B \cup C) + P(A \cap B^c \cap C^c) = P(B) + P(C) - P(B \cap C) + P(A \cap B^c \cap C^c) = 0.7$ $\Rightarrow P((A \cup B \cup C)^c) = P(A^C \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 0.3$
   Since $B^c \cap C^c$ is the disjoint union of $A^c \cap B^c \cap C^c$ and $A \cap B^c \cap C^c$, we have that $P(B^c \cap C^c) =$ $P(A^c \cap B^c \cap C^c) + P(A \cap B^c \cap C^c) = 0.1 + 0.3 = 0.4$.
   Finally, $P(A^c \cap B)$ cannot be computed from the given data.

2. (a) If $\mathcal{X}$ is a Poisson random variable with mean 4, what is $\mathsf{var}(2 + 3\mathcal{X})$?
   Solution: Since the Poisson parameter λ equals $\mathsf{E}[\mathcal{X}]$, and $\mathsf{var}(\mathcal{X}) = \lambda$, we get that $\mathsf{var}(\mathcal{X}) = 4$, and $\mathsf{var}(2 + 3\mathcal{X}) = 3^2 \cdot \mathsf{var}(\mathcal{X}) = 36$. In more detail, $\mathsf{E}[2 + 3\mathcal{X}] = 2 + 3 \cdot \mathsf{E}[\mathcal{X}] = 14$, while $\mathsf{E}[(2 + 3\mathcal{X})^2] = 4 + 12 \cdot \mathsf{E}[\mathcal{X}] + 9 \cdot \mathsf{E}[\mathcal{X}^2] = 4 + 48 + 9 \cdot (4 + 4^2) = 232$, giving $\mathsf{var}(2 + 3\mathcal{X}) = \mathsf{E}[(2 + 3\mathcal{X})^2] - (\mathsf{E}[2 + 3\mathcal{X}])^2 = 232 - 14^2 = 232 - 196 = 36$ as before.

   (b) Let $\mathcal{Y}$ be a negative binomial random variable with parameters (n, p) where n is known, but the value of p is unknown. It is observed that $\{\mathcal{Y} = k\}$. What is the maximum-likelihood estimate $\hat{p}$ of the parameter p?
   Solution: Note that $k \geq n$. The likelihood function is $\binom{k-1}{n-1} p^n (1-p)^{k-n}$ which, regarded as a function of p, has a maximum at n/k. In more detail, the derivative of $p^n(1-p)^{k-n}$ is

$np^{n-1}(1-p)^{k-n} - p^n(k-n)(1-p)^{k-n-1}$ which has value 0 at $p = n/k$. Thus the likelihood function has an extremum at $p = n/k$, and since the function is 0 at $p = 0$ and at $p = 1$, and positive in between, this extremum is a maximum. Thus, $\hat{p} = n/k$.

3. (a) [8 points] Dilbert has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with $P(H) = p \neq \frac{1}{2}$. The probability that a coin chosen at random from his pocket will land Tails is $\frac{7}{12}$. What is the value of p?

   Solution: $\frac{7}{12} = P(\text{Tails}) = P(\text{Tails}|\text{fair})\frac{2}{3} + P(\text{Tails}|\text{biased})\frac{1}{3} = \frac{1}{2} \times \frac{2}{3} + (1-p) \times \frac{1}{3} \Rightarrow p = \frac{1}{4}$.

   (b) [18 points] Let A and B denote events defined on a sample space. Given that $P(A) = 3/5, P(B) = 2/5$, and $P(B|A) = 1/3$, find $P(A|B), P(A^c \cup B^c)$ and $P(B^c|A^c)$.

   Solution: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)} = \dfrac{(1/3)(3/5)}{(2/5)} = \dfrac{1}{2}$. Note that $P(AB) = \dfrac{1}{5}$.

   $P(A^c \cup B^c) = P((AB)^c) = 1 - P(AB) = \dfrac{4}{5}$.

   $P(A \cup B) = P(A) + P(B) - P(AB) = \dfrac{4}{5}$. Hence, $P(A^cB^c) = 1 - P(A \cup B) = \dfrac{1}{5}$, and $P(B^c|A^c) = \dfrac{P(A^cB^c)}{P(A^c)} = \dfrac{P(A^cB^c)}{1-P(A)} = \dfrac{1/5}{2/5} = \dfrac{1}{2}$.

4. [24 points] At the Democratic National Convention, Hillary Clinton and Barack Obama have equal numbers of delegates committed to them, and neither candidate can win the nomination on a ballot. In desperation, the Convention decides that the two candidates shall debate each other and the winner shall be the nominee of the Democratic Party. On a debate, Clinton wins (event H) with probability $P(H)$, and Obama wins (event B) with probability $P(B)$. A draw (event D) occurs (that is, neither wins) with probability $P(D) = 1 - P(H) - P(B) > 0$. In case of a draw, another debate is held. Successive debates can be regarded as independent trials, and continue until either event H or B occurs, and the Democratic nominee is chosen.

   Express the answers to the following questions in terms of $P(H)$ and $P(B)$.

   (a) [8 points] What is the probability that Hillary is the Democratic nominee?

   Solution: $P(\text{Hillary wins}) = P(H) + P(D)P(H) + \cdots (P(D))^k P(H) + \cdots = P(H)\sum_{k=0}^{\infty}(P(D))^k$

   $= P(H)\dfrac{1}{1-P(D)} = \dfrac{P(H)}{P(H)+P(B)}$.

   (b) [8 points] Given that no more than 5 debates were held, what is the conditional probability that Obama won the nomination?

   Solution: The intersection of the events {Obama wins nomination} and {no more than 5 debates held} is $\{B, DB, DDB, DDDB, DDDDB\}$ which has probability

   $P(B) + P(D)P(B) + (P(D))^2P(B) + (P(D))^3P(B) + (P(D))^4P(B) = P(B)\dfrac{1-(P(D))^5}{1-P(D)}$. Now, the probability of the event {no more than 5 debates held} is clearly

   $1 - P(\text{first 5 debates are draws}) = 1 - (P(D))^5$. Hence,

   $P\{\text{Obama wins nomination}|\text{at most 5 debates}\} = \dfrac{P(B)\frac{1-(P(D))^5}{1-P(D)}}{1-(P(D))^5} = \dfrac{P(B)}{1-P(D)}$

   $= \dfrac{P(B)}{P(H)+P(B)}$ which is just the *unconditional* probability that he wins the nomination.

   (c) [8 points] What is the expected number of debates at the Democratic National Convention?

   Solution: On each debate, either the event D occurs and the next debate in the series is scheduled, or the event $H \cup B$ occurs in which case the series of debates ends. Obviously, the number of debates is a *geometric* random variable with parameter $p = P(H) + P(B)$, and hence the expected number of debates is $p^{-1} = \dfrac{1}{P(H)+P(B)}$.