University of Illinois Spring 2008

ECE 313: Solutions to Hour Exam II

1. [33 points] The received signal $\mathcal{R}$ in a digital communication system corresponds to one of two hypotheses:

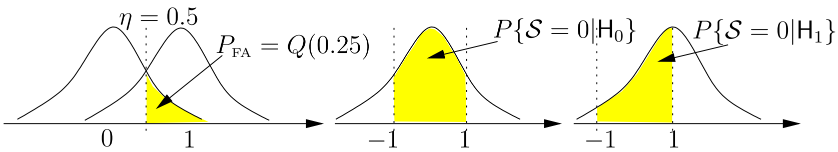
$$\begin{array}{ll} \mathsf{H}_0: & \mathcal{R} \sim \mathsf{Gaussian}(0, 2^2) \\ \mathsf{H}_1: & \mathcal{R} \sim \mathsf{Gaussian}(1, 2^2) \end{array}$$

   (a) [10 points] The maximum-likelihood decision rule can be stated in the form

   "Decide that H_1 is the true hypothesis if and only if $\mathcal{R} > \eta$."

   What is the value of η, and what is the value of P_{FA}, the false-alarm probability (or probability of Type I error) of the maximum-likelihood decision rule?

   Solution: The ML threshold is where the two pdfs cross: $\eta = 0.5$.



   $P_{\text{FA}} = P\{\mathcal{R} > 0.5 | \mathsf{H}_0\} = Q\left(\frac{0.5}{2}\right) = Q(0.25) = 1 - \Phi(0.25) = 1 - 0.5987 = 0.4013.$

   (b) [18 points] Now suppose that the received signal $\mathcal{R}$ is passed through a quantizer whose output $\mathcal{S}$ is

$$\mathcal{S} = \begin{cases} -\alpha, & \text{if } \mathcal{R} < -1, \\ 0, & \text{if } -1 \leq \mathcal{R} < +1, \\ +\alpha, & \text{if } \mathcal{R} \geq +1. \end{cases}$$

   and the decision is made based on the value of $\mathcal{S}$.

   Complete the likelihood matrix shown below, and indicate on it the maximum-likelihood decision rule by shading entries.

   Solution: From the middle figure above, $P\{\mathcal{S} = 0|\mathsf{H}_0\} = \Phi(1/2) - \Phi(-1/2) = 0.3830$, while $P\{\mathcal{S} = +\alpha|\mathsf{H}_0\} = P\{\mathcal{S} = -\alpha|\mathsf{H}_0\} = 1 - \Phi(1/2) = 0.3085$.

   From the right-hand figure above, $P\{\mathcal{S} = 0|\mathsf{H}_1\} = \Phi(0) - \Phi(-1) = 0.3413$, $P\{\mathcal{S} = +\alpha|\mathsf{H}_1\} = 0.5, P\{\mathcal{S} = -\alpha|\mathsf{H}_1\} = \Phi(-1) = 1 - \Phi(1) = 0.1587$. Hence,

	$\mathcal{S} = -\alpha$	$\mathcal{S} = 0$	$\mathcal{S} = +\alpha$
H_0	0.3085	0.3830	0.3085
H_1	0.1587	0.3413	0.5000

   (c) [5 points] For the maximum-likelihood decision rule of part (b), what is P_{FA}?

   Solution: P_{FA} = sum of unshaded entries on H_0 row $= P\{\mathcal{S} = +\alpha | \mathsf{H}_0\} = 0.3085$

2. [32 points]

   (a) [10 points] $\mathcal{W}$ denotes a *uniform* random variable with mean 1 and variance 3. Find $P\{\mathcal{W} < 0\}$.

   Solution: If $\mathcal{W} \sim U[a, b]$, then $\mathsf{E}[\mathcal{W}] = (a + b)/2 = 1$ while $\mathsf{var}(\mathcal{W}) = (b - a)^2/12 = 3$ giving that $(b - a)^2 = 36$, i.e., $b - a = 6$ from which we get that $a = -2, b = 4$. Hence, $P\{\mathcal{W} < 0\} = 1/3$.

(b) [11 points] Suppose $\mathcal{X}$ is an exponential random variable with parameter λ. Calculate the pdf of the random variable $\mathcal{Y} = \sqrt{2\lambda\mathcal{X}}$.

Solution: Since $\mathcal{X}$ is non-negative, so is $\mathcal{Y}$. So for any $v \geq 0$, we have:
$F_{\mathcal{Y}}(v) = P(\mathcal{Y} \leq v) = P\left(\sqrt{2\mathcal{X}\lambda} \leq v\right) = P\left(\mathcal{X} \leq \frac{v^2}{2\lambda}\right) = F_{\mathcal{X}}\left(\frac{v^2}{2\lambda}\right) = 1 - \exp\left(-\frac{v^2}{2}\right)$
$\Rightarrow f_{\mathcal{Y}}(v) = v\exp\left(-\frac{v^2}{2}\right)$ Thus for all v we have that: $f_{\mathcal{Y}}(v) = \begin{cases} v\exp\left(-\frac{v^2}{2},\right) & v \geq 0, \\ 0, & \text{otherwise.} \end{cases}$
This is called a *Rayleigh density function.*

(c) [11 points] Suppose that $\mathcal{Z}$ is a standard Gaussian random variable. Find $\mathsf{E}[|\mathcal{Z}|]$.

Solution:
By LOTUS, $\mathsf{E}[|\mathcal{Z}|] = \int_{-\infty}^{\infty} |v|\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv = 2\int_0^{\infty} v\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)dv = \sqrt{\frac{2}{\pi}}$
since $v\exp(-v^2/2), v \geq 0$ is a valid pdf (cf. part (b) above!)

3. [35 points] Consider a Poisson process with arrival rate λ. Let $N(a, b]$ denote the number of arrivals in the time interval $(a, b]$, and let $\mathcal{X}_1$ denote the time of the first arrival *after* $t = 0$.

(a) [6 points] *State* the pmf of $N(0, \tau]$ and the pdf of $\mathcal{X}_1$. No derivation need be provided.
Solution: $N(0, \tau]$ is a Poisson random variable with parameter $\lambda\tau$, and thus
$P\{N(0, \tau] = k\} = \exp(-\lambda\tau)\frac{(\lambda\tau)^k}{k!}, k = 0, 1, 2, \ldots$
$\mathcal{X}_1$ is an exponential random variable with parameter $\lambda \Rightarrow f_{\mathcal{X}_1}(u) = \begin{cases} \lambda\exp(-\lambda u), & u \geq 0 \\ 0, & \text{otherwise.} \end{cases}$
Note that $P\{\mathcal{X}_1 > \tau\} = \exp(-\lambda\tau)$.

(b) [12 points] The event $A = \{N(0, \tau] = 0\}$ is the same as the event $B = \{\mathcal{X}_1 > \tau\}$.
According to your answers of part (a), what is $P\{N(0, \tau] = 0\}$? and what is $P\{X_1 > \tau\}$? Does $P(A)$ equal $P(B)$?
Solution:
Yes, $P(A) = P\{N(0, \tau] = 0\} = \exp(-\lambda\tau)$ equals $P(B) = P\{X_1 > \tau\} = \exp(-\lambda\tau)$.

(c) [10 points] Now let T denote a fixed positive number and n a positive integer. Let τ be such that $0 < \tau < T$. If C denotes the event $\{N(0, T] = n\}$, what is the value of $P(B|C)$?
Hint: Express $P(B \cap C) = P(A \cap C)$ in terms of $N(0, \tau]$ and $N(\tau, T]$.
Solution: $P(B|C) = \frac{P(BC)}{P(C)} = \frac{P(AC)}{P(C)} = \frac{P\{\{N(0,\tau] = 0\} \cap \{N(\tau, T] = n\}\}}{P\{N(0, T] = n\}}$
$= \frac{P\{N(0,\tau] = 0\}P\{N(\tau, T] = n\}}{P\{N(0, T] = n\}} = \frac{\exp(-\lambda\tau)\exp(-\lambda(T-\tau))(\lambda(T-\tau))^n/n!}{\exp(-\lambda T)(\lambda T)^n/n!} = \left(1 - \frac{\tau}{T}\right)^n.$

(d) [7 points] Determine the *conditional* pdf of $\mathcal{X}_1$ given that the event C occurred, that is, given that n arrivals occurred in the time interval $(0, T]$.
Solution: From part (c), we have that $P(B|C) = P\{\mathcal{X}_1 > \tau|C\} = (1-\tau/T)^n$, and hence, $F_{\mathcal{X}_1|C}(\tau|C)$, the conditional CDF of $\mathcal{X}_1$ given that event C has occurred is $1-(1-\tau/T)^n$
for $0 < \tau < T$. Hence, $f_{\mathcal{X}_1|C}(\tau|C) = \begin{cases} \frac{n}{T}\left(1 - \frac{\tau}{T}\right)^{n-1}, & 0 < \tau < T, \\ 0, & \text{otherwise.} \end{cases}$