University of Illinois Spring 2007

ECE 413: Solutions to Problem Set 6

1. (a) The maximum-likelihood (ML) decision rule is indicated by the shading below.

Hypothesis	$\mathcal{Y}=1$	$\mathcal{Y}=2$	$\mathcal{Y}=3$
H_1: s_1 transmitted	0.80	0.10	0.10
H_2: s_2 transmitted	0.05	0.90	0.05
H_3: s_3 transmitted	0.02	0.08	0.90

(b) Since the hypotheses are equally likely, the MAP decision rule is the same as the ML decision rule! The joint probability matrix is as shown below, and the shading indicates the MAP (and the ML) decision rule.

Hypothesis	$\mathcal{Y}=1$	$\mathcal{Y}=2$	$\mathcal{Y}=3$
H_1: s_1 transmitted	$\frac{0.80}{3}$	$\frac{0.10}{3}$	$\frac{0.10}{3}$
H_2: s_2 transmitted	$\frac{0.05}{3}$	$\frac{0.90}{3}$	$\frac{0.05}{3}$
H_3: s_3 transmitted	$\frac{0.02}{3}$	$\frac{0.08}{3}$	$\frac{0.90}{3}$

Both rules have error probability equal to the sum of the unshaded entries in the joint probability matrix, giving

$$P_e = \frac{0.10+0.10+0.05+0.05+0.02+0.08}{3} = \frac{0.4}{3} = \frac{2}{15}.$$

(c) When the prior probabilities are $\pi_1 = 0.1, \pi_2 = 0.3, \pi_3 = 0.6$, the joint probability matrix is as shown below with the MAP decision rule indicated by shading. The ML decision rule is shown by underlines and is the same as the MAP rule

Hypothesis	$\mathcal{Y}=1$	$\mathcal{Y}=2$	$\mathcal{Y}=3$
H_1: s_1 transmitted	$\underline{0.080}$	0.010	0.010
H_2: s_2 transmitted	0.015	$\underline{0.270}$	0.015
H_3: s_3 transmitted	0.012	0.048	$\underline{0.540}$

The error probability for the MAP decision rule is the sum of the unshaded entries, while for the ML decision rule, it is the sum of the entries with no underlines. It is easy to get $P_{e,\text{MAP}} = P_{e,\text{ML}} = 1-(0.08+0.27+0.54) = 0.11$.

2. (a) The ML decision rule is indicated by shading in the likelihood matrix below.

Hypothesis	$\mathcal{X}=3$	$\mathcal{X}=6$	$\mathcal{X}=9$	$\mathcal{X}=12$
H_e: excellent	0.02	0.08	0.15	0.75
H_g: good	0.10	0.15	0.60	0.15
H_a: average	0.20	0.65	0.10	0.05

(b) $P(\text{excellent student is labeled as good}) = P(\mathcal{X}=9|\mathsf{H}_e) = 0.15$.
$P(\text{excellent student is labeled as average}) = P(\{\mathcal{X}=6\}\cup\{\mathcal{X}=3\}|\mathsf{H}_e) = 0.02+0.08 = 0.1$.
$P(\text{average student is misclassified}) = P(\{\mathcal{X}=9\}\cup\{\mathcal{X}=12\}|\mathsf{H}_a) = 0.15$.

(c) The conditional error probabilities of the ML decision rule are $P(E|\mathsf{H}_e) = 0.25$, $P(E|\mathsf{H}_g) = 0.4$, $P(E|\mathsf{H}_a) = 0.15$. Hence, the error probability is

$$P(E) = P(E|\mathsf{H}_e)\pi_e + P(E|\mathsf{H}_g)\pi_g + P(E|\mathsf{H}_a)\pi_a = 0.05+0.22+.0375 = 0.3075.$$

(d) The joint probability matrix is as shown below together with the MAP decision rule.

Hypothesis	$\mathcal{X}=3$	$\mathcal{X}=6$	$\mathcal{X}=9$	$\mathcal{X}=12$
H_e: excellent	0.0040	0.0160	0.0300	0.1500
H_g: good	0.0550	0.0825	0.3300	0.0825
H_a: average	0.0500	0.1625	0.0250	0.0125

$P(E) = 1 - (0.15 + 0.33 + 0.1625 + 0.055) = 0.3025$ which is slightly smaller than that of the ML decision rule. But note that students getting D's are classified as good while students getting C's are classified as average. Holy capricious grading complaint, Batman! This anomalous result is due to the fact that there are so many good students that more than half the students getting D's are good students. Moral: Good students should work hard in ECE 413 to avoid D's...

(e) Now the joint probability matrix looks as shown below, and all students are classified as excellent regardless of their course grade!

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_e: excellent	0.0190	0.0760	0.1425	0.7125
H_g: good	0.0050	0.0075	0.0300	0.0075
H_a: average	0.0000	0.000	0.0000	0.0000

Obviously, there is no need for homework, quizzes, and exams at the Lake Wobegon campus since the results are ignored anyway.

Noncredit exercise: Write a letter to the Governor asking him to demand that in view of the excellence of the UIUC student body, the University must adopt the Lake Wobegon approach and eliminate all work in courses as a cost-cutting measure ...

3. (a) If $\mathcal{X} = n$, the likelihood ratio has value

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1}{p_0}\left(\frac{1-p_1}{1-p_0}\right)^{n-1} > 1 \text{ if } (n-1)\ln\left(\frac{1-p_1}{1-p_0}\right) > \ln\left(\frac{p_0}{p_1}\right).$$

Since $p_1 < p_0$, we have that $1-p_1 > 1-p_0$ and $\ln((1-p_1)/(1-p_0)) > 0$. Therefore, the maximum likelihood decision rule is

$$\text{“Decide that } \mathsf{H}_1 \text{ is the true hypothesis if } \mathcal{X} > 1 + \frac{\ln\left(\frac{p_0}{p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}\text{”}$$

(b) The MAP decision rule chooses H_1 as the true hypothesis if the likelihood ratio exceeds the threshold π_0/π_1. Now $\Lambda(1) = p_1/p_0 < 1$. Since $1-p_1 > 1-p_0$, we see that

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1(1-p_1)^{n-2}}{p_0(1-p_0)^{n-2}}\left(\frac{1-p_1}{1-p_0}\right) = \Lambda(n-1)\left(\frac{1-p_1}{1-p_0}\right) > \Lambda(n-1),$$

and so $\Lambda(1) = p_1/p_0$ is the smallest value of the likelihood ratio. It follows that if $\pi_0/\pi_1 = \pi_0/(1-\pi_0) < p_1/p_0$, that is, if $\pi_0 < p_1/(p_0+p_1)$, the MAP decision rule will always decide that H_1 is the true hypothesis regardless of the observed value of $\mathcal{X}$.

On the other hand, since $\Lambda(n)$ increases monotonically without bound as n increases, there is *no* value of $\pi_0 < 1$ for which π_0/π_1 can be guaranteed to be larger than the likelihood ratio no matter what value $\mathcal{X}$ takes on.

4. (a) $p_0(86) = \binom{90}{86}(0.9)^{86}(0.1)^4$, $\quad p_1(86) = \binom{90}{71}(0.9)^{71}(0.1)^{19}$. The likelihood ratio is $p_1(86)/p_0(86) \approx 0.027$ and hence the ML decision is that the connecting flight is late.

(b) Repeat part (a) for the case when the gate agent observes that $\mathcal{X} = 96$. Obviously $p_0(96) = 0$, while $p_1(96) = \binom{90}{81}(0.9)^{81}(0.1)^9$. The likelihood ratio is $p_1(86)/p_0(86) = \infty$ and hence the ML decision is that the connecting flight is on time. In fact, it should be obvious that for $\mathcal{X} > 90$, the ML decision is that the connecting flight is on time.

(c) The MAP decision rule is the same as the ML decision rule.