University of Illinois **Spring 2007**
ECE 413: Problem Set 13 **Due 5/2/07 at beginning of class**

Jointly Gaussian random variables, minimum mean square error estimation, bounds and limit theorems, and Markov chains

Assigned reading: *Ross*, Sections 7.2, 7.3, 7.4.1-7.4.3, 7.5, 8-1-8.3, 9.2
Noncredit exercises: Chapter 7, Problems 30,33,34,3,37,38,40,45,48,50,51 and Theoretical Exercises 1,2,17,22,23,38-43; Chapter 8, Problems 1-9, 15. Chapter 9, Theoretical Exercises 4-8.
Reminder: Final Exam: The exam will be held Tuesday, May 8, 1:30-4:30 p.m., in Room 269 Everitt Laboratory. The exam is comprehensive, although about half of the exam will involve material covered since the second exam (i.e. material related to problem sets 11-13). You may bring three 8.5" by 11" sheets of notes to the final exam. You may use both sides of the sheets, using font size 10 or larger (or similar handwriting size). The exam is closed book otherwise. Calculators, laptop computers, tables of integrals, etc. are not permitted.

1. A linear minimum mean square error estimator
Let $X = Y + N$, where Y has the exponential distribution with parameter λ, and N is Gaussian with mean 0 and variance σ^2. The variables Y and N are independent, and the parameters λ and σ^2 are strictly positive. (Recall that $E[Y] = \frac{1}{\lambda}$ and $\mathrm{Var}(Y) = \frac{1}{\lambda^2}$.)
(a) Find L, the LMMSE estimator of Y given X, and also find the resulting mean square error.
(b) It can be shown that if nonlinear functions of Y are permitted, then an estimator of Y given X can be found with a strictly smaller mean square error. Show how to find one. (Hint: You don't need to find the optimal estimator, just one better than L. Can Y be negative? Can L be negative?)

2. Working with joint Gaussians
Let X and Y be jointly Gaussian random variables with mean zero, $\sigma_X^2 = 5$, $\sigma_Y^2 = 2$, and $\mathrm{Cov}(X, Y) = -1$. Find $P\{X + 2Y \geq 1\}$.

3. Simulating joint Gaussians
Together, the two parts of this problem illustrate how to simulate a pair of jointly Gaussian random variables with specified parameters. (a) Explain how a pair of independent unit normal random variables (A, B) can be generated using two independent random variables U and V, each uniform on the interval $[0, 1]$. (Hint: Think polar coordinates.)
(b) Let (A, B) be as in part (a), and let $X = aA + bB$ and $Y = cA + dB$ for some constants a, b, c, d. Under what conditions on the constants do X and Y have the joint distribution described in the previous problem? Find a numerical value of the constants satisfying these conditions.

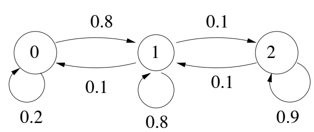
4. Estimating the square of a joint Gaussian
Let X and Y be jointly Gaussian random variables with mean zero, with $\sigma_X^2 = \sigma_Y^2 = 1$, and $\mathrm{Cov}(X, Y) = \rho$. Find $E[Y^2|X]$, the best estimator of Y^2 given X. (Hint: X and Y^2 are not jointly Gaussian. But you know the conditional distribution of Y given $X = x$ (see end of Section 7.5 of Ross) and can use it to find the conditional second moment of Y given $X = x$.)

5. Weak law of large numbers for slightly correlated variables
Suppose $X_1, \ldots, X_{100}$ are random variables, each with mean $\mu = 5$ and variance $\sigma^2 = 1$. Suppose also that $|\mathrm{Cov}(X_i, X_j)| \leq 0.1$ if $i = j \pm 1$, and $\mathrm{Cov}(X_i, X_j) = 0$ if $|i - j| \geq 2$. Let $S_{100} = X_1 + \cdots + X_{100}$.
(a) Show that $\mathrm{Var}(S_{100}) \leq 120$.
(b) Use part (a) and Chebychev's inequality to find an upper bound on $P(|\frac{S_{100}}{100} - 5| \geq 0.5)$.

6. Normal approximation for quantization error
Suppose each of 100 real numbers are rounded to the nearest integer and then added. Assume the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. Using the normal approximation suggested by the central limit theorem, find the approximate probability that the absolute value of the sum of the errors is greater than 5.

7. A comparison of bounds and the normal approximation
A single fair die with the numbers 1 through 6 on it is rolled repeatedly. Let X_i denote the number appearing on the i^{th} roll. Assume the X_i's are independent, and let $S_n = X_1 + X_2 + \ldots + X_n$.
(a) Use Markov's inequality to find an upper bound on $P[S_{100} \geq 400]$.
(b) Use Chebyshev's inequality to find an upper bound on $P[S_{100} \geq 400]$. (Either use the two-sided version and the fact a one-sided tail probability can be bounded by a two-sided tail probability, or use the one-sided version to get a tighter bound.)
(c) Compute the approximation to $P[S_{100} \geq 400]$ suggested by the central limit theorem.

8. A three state Markov chain
Consider a three state Markov chain with the one-step transition probability diagram shown.



Such a chain could arise from a model of one-dimensional motion on the integers. State 0 could mean take a step left, state 1 take no step, and state 2 take a step right.
(a) Give the corresponding 3×3 transition probability matrix $P = (P_{i,j} : 0 \leq i \leq 2, 0 \leq j \leq 2)$.
(b) Find $(P_{10}^{(4)}, P_{11}^{(4)}, P_{12}^{(4)})$ (This is the distribution of the state after four steps, for initial state 1, and it is also the middle row of P^4, which can be computed by squaring P and then squaring the result.
(c) Find the equilibrium probability vector $\Pi = (\Pi_0, \Pi_1, \Pi_1)$.