University of Illinois **Spring 2007**
ECE 413: Problem Set 12

Functions of random variables; conditional pdfs, covariance

Due: 4/25/07 at the beginning of class

Reading: Chapters 6 and 7 of the textbook

Noncredit Exercises: Chapter 6, Problems 26, 28-30, 41-43, 51, 54; Theoretical Exercises: 8, 14, 22, 23, 33;
Chapter 7: Problems 1, 16, 26, 29, 34, 36; Theoretical Exercises: 1, 2, 17, 22, 23, 40

Problems:

1. We return to the "random chord" of Problem 3 of Problem Set 10. Yet another way of defining a "random chord" is to assume the midpoint of the chord to be anywhere inside the circle of radius 1 with equal probability. The chord is, of course, perpendicular to the diameter that passes through the chosen point. Thus, let the random point $(\mathcal{X}, \mathcal{Y})$ be *uniformly distributed* on the interior of the circle of unit radius centered at the origin (this region is called the unit disc — nomenclature that might be familiar to DSPists).
   (a) Find the probability that the length $\mathcal{L}$ of the random chord is greater than the side of the equilateral triangle inscribed in the circle.
   (b) Express $\mathcal{L}$ as a function of the random variable $(\mathcal{X}, \mathcal{Y})$ and find the probability density function for $\mathcal{L}$.
   (c) Find the average length of the chord, i.e. find $\mathsf{E}[\mathcal{L}]$.

2. The joint pdf $f_{\mathcal{X},\mathcal{Y}}(u,v)$ of two jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ is said to have *circular symmetry* if the joint pdf can be expressed as a function of $\sqrt{u^2+v^2}$, that is, $f_{\mathcal{X},\mathcal{Y}}(u,v) = g\left(\sqrt{u^2+v^2}\right) = g(r)$ where $r = \sqrt{u^2+v^2}$. Now, consider a joint pdf with circular symmetry for which $g(r) = \begin{cases} c(2-r), & 0 < r < 2, \\ 0, & \text{otherwise.} \end{cases}$
   (a) Sketch the pdf and find the value of c.
   (b) Let $\mathcal{R}$ denote the distance of the random point $(\mathcal{X}, \mathcal{Y})$ from the origin. Find the pdf of $\mathcal{R}$.

3. Let $\mathcal{W} = \max\{\mathcal{X}, \mathcal{Y}\}$ and $\mathcal{Z} = \min\{\mathcal{X}, \mathcal{Y}\}$ be the maximum and minimum of two random variables $\mathcal{X}$ and $\mathcal{Y}$.
   (a) For each (α, β), $-\infty < \alpha < \infty, -\infty < \beta < \infty$, express $F_{\mathcal{W},\mathcal{Z}}(\alpha, \beta)$ in terms of $F_{\mathcal{X},\mathcal{Y}}(u,v)$.
   (b) Simplify your answer to part (a) for the case when $\mathcal{X}$ and $\mathcal{Y}$ are *independent* random variables.
   (c) From your answer to part (b), deduce the joint pdf of $\mathcal{W}$ and $\mathcal{Z}$ for the case when $\mathcal{X}$ and $\mathcal{Y}$ are *continuous* independent random variables.

4. [Unbelievable but true; this problem is a *lot* easier than it looks if you follow the instructions carefully.]

   Let $\mathcal{X}$ denote a *zero-mean* Gaussian random variable with variance σ^2. Use the formula of Example 7b of Section 5.7 of Ross (p. 243) to show that $\mathcal{X}^2$ is a gamma random variable with parameters $(\frac{1}{2}, \frac{1}{2\sigma^2})$. (The value of $\Gamma(\frac{1}{2})$ can be found in Problem 20 of Chapter 5.

   (a) Next, suppose that $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are *independent* zero-mean Gaussian random variable with variance σ^2. What are the pdfs of $\mathcal{X}^2$, $\mathcal{Y}^2$, and $\mathcal{Z}^2$? Why are $\mathcal{X}^2$, $\mathcal{Y}^2$, and $\mathcal{Z}^2$ independent random variables? Explain briefly.

   (b) Use the result of Proposition 3.1 of Chapter 6 of Ross to *state* what is the *type* of pdf of $\mathcal{W} = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$, and *write down* explicitly the pdf of $\mathcal{W}$.

   (c) Prove that $\mathsf{E}[\mathcal{W}] = 3\sigma^2$. If you actually evaluated an integral instead of using LOTUS, shame on you!

   (d) In a physical application, $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $\mathcal{H} = \frac{1}{2}m\mathcal{W}$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the *average kinetic energy* is $\mathsf{E}[\mathcal{H}] = \mathsf{E}[\frac{1}{2}m\mathcal{W}] = \frac{1}{2}m\mathsf{E}[\mathcal{W}] = \frac{3}{2}m\sigma^2 = \frac{3}{2}kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in degrees Kelvin. (Note that the average energy is $\frac{1}{2}kT$ *per dimension.*) Show that the kinetic energy $\mathcal{H}$ has the *Maxwell-Boltzmann pdf* $f_{\mathcal{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-\frac{3}{2}}\sqrt{\beta}\exp(-\beta/kT), \beta > 0$.

   (e) $\mathcal{V} = \sqrt{\mathcal{W}} = \sqrt{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}$ is the speed of the molecule. Show that the pdf of $\mathcal{V}$ is $f_{\mathcal{V}}(\gamma) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right), \gamma > 0$. (cf. Theoretical Exercise 1 of Chapter 5).

5. Consider the random point $(\mathcal{X}, \mathcal{Y})$ whose joint pdf is uniformly distributed on the region

$$\{(u, v) \ : \ 0 < u < 1, 0 < v < 1, \max\{u, v\} > 1/2\}.$$

   (a) Find the marginal pdf of $\mathcal{X}$.

   (b) Find $\mathsf{E}[\mathcal{X}]$ and $\mathsf{var}(\mathcal{X})$.

   (c) Explain why the random variable $\mathcal{Y}$ has the same mean and variance as $\mathcal{X}$.

   (d) Compute $\mathsf{E}[\mathcal{X}\mathcal{Y}]$ and hence determine $\mathsf{cov}(\mathcal{X}, \mathcal{Y})$.

   (e) Compute the pdf of the random variable $\mathcal{Z} = \mathcal{Y}/\mathcal{X}$.

   (f) Compute the *conditional pdf* $f_{\mathcal{X}|\mathcal{Y}}(u|v)$ for $0 < v < \frac{1}{2}$ and for $\frac{1}{2} < v < 1$. Be sure to specify the ranges of u for which your expressions are valid and verify that the conditional pdfs that you obtain satisfy the standard properties of pdfs.

   (g) Use the theorem of total probability to compute the unconditional pdf of $\mathcal{X}$ from the conditional pdfs of part (f) and compare your answer to the pdf of $\mathcal{X}$ that you found in part (a).