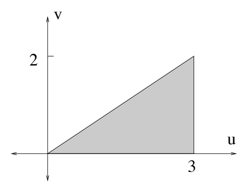
University of Illinois **Spring 2007**
ECE 413: Problem Set 11 **Due 4/18/07 at beginning of class**

Jointly distributed random variables

Assigned reading: Ross, Sections 6.1-6.5
Noncredit exercises: Ross, Chapter 6: problems 1-23, theoretical exercises 1-17, self-test 1-15.

1. The CDF for a uniform distribution within a triangle

Suppose the random point (X, Y) is uniformly distributed over the triangular region shown.



(a) Find the joint pdf of X and Y. Be sure to specify $f_{X,Y}(u, v)$ for all (u, v) in the two-dimensional plane, and not just for points inside the triangle.
(b) In a sketch of the entire (u_o, v_o) plane, indicate regions where the joint CDF $F_{X,Y}(u_o, v_o)$ is zero, where it is one, where it is a function of u_o only, and where it is a function of v_o only.
(c) Complete the following equation for the joint CDF:

$$F_{XY}(u_o, v_o) = \begin{cases} 0 & \text{if ???} \\ ??? & \text{if } 0 \leq u_o \leq 3 \text{ and } 3v_o \leq 2u_o \\ ??? & \text{if ???} \\ ??? & \text{if ???} \\ 1 & \text{if ???} \end{cases}$$

2. A joint pmf with geometric distribution marginals

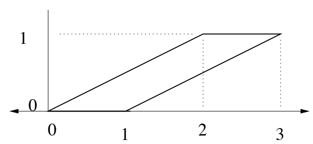
Suppose a plant tests wafers one at a time, exposing each wafer to phase one testing, and exposing each wafer passing phase one testing to phase two testing. Wafers failing phase one testing are discarded. Let p_0 denote the probability a wafer does not pass phase one, let p_1 denote the probability a wafer passes phase one but not phase two, and let p_2 denote the probability a wafer passes both phases. Note that $p_0 + p_1 + p_2 = 1$. Let X be the number of wafers tested until one passes both phases, and let Y denote the number of the first X wafers that pass phase one (so the count Y includes the first wafer to pass both phases).
(a) Explain, using little or no calculation, why Y has a geometric distribution, and identify the parameter of the distribution.
(b) Find the joint pmf of X and Y.
(c) Using your answer to part (b) and summation of a series, find the marginal pmf, p_Y. This gives a second derivation of the fact that Y has a geometric distribution.

3. Uniform distribution on a rotated square

Suppose the random point (X, Y) is uniformly distributed over the square region with corners at the points: $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.
(a) Calculate the marginal pdfs of X and Y. Are X and Y independent?
(b) Compute $E[X]$ and $\mathrm{Var}(X)$.
(c) Calculate the pdf of $A = X + Y$. (Hint: First find the CDF and differentiate.)
(d) Calculate the pdf of $C = X/Y$. (Hint: First find the CDF and differentiate.)

4. Uniform distribution over a parallelogram

Let the random variables X and Y be jointly uniformly distributed over the region shown.



(a) Determine the value of f_{XY} on the region shown.
(b) Find the pdf of X.
(c) Find the mean and variance of X.
(d) Find the conditional pdf of Y given that $X = a$, for $0 \leq a \leq 1$
(e) Find the conditional pdf of Y given that $X = a$, for $1 \leq a \leq 2$
(f) Find and sketch $E[Y|X = a]$ as a function of a. Be sure to specify the values of a for which this conditional expectation is well defined.

5. Functions of independent exponential random variables

Let X_1 and X_2 be independent random varibles, with X_i being exponentially distributed with parameter λ_i.
(a) Find the CDF and pdf of $Z = \min\{X_1, X_2\}$.
(b) Find the CDF and pdf of $R = \frac{X_1}{X_2}$.