University of Illinois **Spring 2007**
ECE 413: Problem Set 10

Function of a random variable; hazard rates, decision-making

Due: 4/4/07 at the beginning of class

Reading: Chapter 5 of the textbook

Noncredit Exercises: Chapter 5, Problems 10-19; 21, 22, 24, 31-41

Problems:

1. The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modelled as a continuous random variable $\mathcal{V}$ with pdf
$$f_{\mathcal{V}}(u) = 0.5 \exp(-|u|), -\infty < u < \infty.$$
Then, the current $\mathcal{I}$ is also a continuous random variable.
   (a) What values can $\mathcal{I}$ take on?
   (b) Find the CDF of $\mathcal{I}$.
   (c) Find and sketch the pdf of $\mathcal{I}$.

2. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.
   (a) Find the pdf of $\mathcal{Y} = \mathcal{X}^2$.
   (b) Find the pdf of $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0 \\ -u^2, & u < 0 \end{cases}$.

3. In Example 3d on page 217, Ross gives one definition of a random chord of a circle as one whose midpoint $\mathcal{X}$ is uniformly distributed on $[0, r]$ where r denotes the radius of the circle. Find the pdf of $\mathcal{Y}$, the length of this random chord.

4. ["Give me an A! Give me a D! Give me a converter! What have we got? An A/D converter! Go Team!"] A signal $\mathcal{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathcal{Y}$ (where $\mathcal{Y} = \alpha$ if $\mathcal{X} > 0$ and $\mathcal{Y} = -\alpha$ if $\mathcal{X} \leq 0$) is used. Note that $\mathcal{Y}$ is a *discrete* random variable.
   (a) What is the pmf of $\mathcal{Y}$?
   (b) The *squared error* in representing $\mathcal{X}$ by $\mathcal{Y}$ is $\mathcal{Z} = \begin{cases} (\mathcal{X} - \alpha)^2, & \text{if } \mathcal{X} > 0, \\ (\mathcal{X} + \alpha)^2, & \text{if } \mathcal{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathcal{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathcal{Z}]$. Use LOTUS to calculate $\mathsf{E}[\mathcal{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathcal{Z}]$.
   (c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathcal{X}$ to the nearest integer $\mathcal{W}$ in the range 3 to +3. Thus, $\mathcal{W} = 3$ if $\mathcal{X} \geq 2.5$, $\mathcal{W} = 2$ if $1.5 \leq \mathcal{X} < 2.5$, $\mathcal{W} = 1$ if $0.5 \leq \mathcal{X} < 1.5$, $\cdots$, $\mathcal{W} = -3$ if $\mathcal{X} < -2.5$. Note that $\mathcal{W}$ is also a discrete random variable. Find the pmf of $\mathcal{W}$.

(d) The output of the A/D converter is a 3-bit 2's complement representation of $\mathcal{W}$. Suppose that the output is $(\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0)$. What is the pmf of $\mathcal{Z}_2$? the pmf of $\mathcal{Z}_1$? the pmf of $\mathcal{Z}_0$? Note that $(1, 0, 0)$ which represents -4 is not one of the possible outputs from this A/D converter.

5. Some systems contain two identical components and have the property that the system works as long as at least one of the components is working. The system is put into operation at $t = 0$. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote the time of failure of the two components, and let $\mathcal{Y}$ denote the time of failure of the system.

   (a) The occurrence of the event $\{Y > T\}$ means that the system is working at time T. Express $\{Y > T\}$ in terms of the events $\{\mathcal{X}_1 > T\}$ and $\{\mathcal{X}_2 > T\}$. Assume that the latter two events are *independent*, and express $P\{Y > T\}$ in terms of the probabilities of these events.

   (b) Now suppose that $P\{\mathcal{X}_1 > T\} = P\{\mathcal{X}_2 > T\} = \exp(-\lambda T)$, that is, $\mathcal{X}_1$ and $\mathcal{X}_2$ are exponential random variables with parameter λ.

      i. Use the result $\mathsf{E}[\mathcal{Y}] = \displaystyle\int_0^\infty P\{\mathcal{Y} > T\}\, dT$ to find $\mathsf{E}[\mathcal{Y}]$, the *average lifetime* of the system. The average lifetime is also known as the mean time before failure (MTBF) or mean time to failure (MTTF) in the reliability literature.

      ii. Find the *median* value of $\mathcal{Y}$ by solving the equation $P\{\mathcal{Y} > T\} = \frac{1}{2}$ for T. The median value is sometimes called the *half-life* of the system.

      iii. Find and sketch the hazard rate of the system.

   (c) Compare your answers of parts (b)(i)-(iii) to the MTBF λ^{-1}, the median lifetime $\lambda^{-1} \ln 2$, and the hazard rate λ for a system with only one component instead of two.

6. If hypothesis H_0 is true, the pdf of $\mathcal{X}$ is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is exponential with parameter 10.

   (a) Sketch the two pdfs.

   (b) State the *maximum-likelihood* decision rule in terms of a threshold test on the *observed value* u of the random variable $\mathcal{X}$ instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ to 1.

   (c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part(b)?

   (d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_0/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable $\mathcal{X}$.

   (e) If $\pi_0 = 1/3$, what is the *average* probability of error of the Bayesian decision rule?

   (f) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by $\mathcal{X}$?

   (g) Show that if $\pi_0 > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by $\mathcal{X}$. What is the average probability of error for the maximum-likelihood rule when $\pi_0 > 2/3$?