University of Illinois **Spring 2007**
ECE 413: Problem Set 5 **Due 2/23/07 at beginning of class (new - due to snow)**

Total probability and Bayes' formula

Assigned reading: *Ross* Sections 3.1–3.3 and first five pages of Section 3.4.
Noncredit exercises: Chapter 3, problems 1,2,5,10,12,16,32,38,39,44,61.

Reminder: Hour exam 1 is on Monday, February 26, 7-8 p.m. in Room 269 Everitt Laboratory. You may bring one sheet of notes, two sided, to the exam.

1. Parity of a binomial random variable, revisited
Suppose a sequence of independent trials is conducted with possible outcomes 0 or 1, with outcome 1 having probability p. Let X_i denote the i^{th} outcome. Let $Y_0 = 0$ and $Y_n = X_1 + \cdots + X_n$ for any $n \geq 1$. Thus, for each n, Y_n is a binomial random variable with parameters n and p. Let $a_n = P(G_n)$, where $G_n = \{Y_n \text{ is even}\}$. Trivially, $a_0 = 1$.
(a) Express G_{n+1} in terms of the events G_n and $\{X_{n+1} = 1\}$.
(b) Use your answer to part (a) to express a_{n+1} in terms of a_n and p.
(c) Solve the recursion to find a_n for $n \geq 1$.

2. Ternary channel
A communication system transmits one of three signals, s_1, s_2, or s_3, each with equal probability. The reception is corrupted by noise, causing the transmission to be changed according to the following table of conditional probabilities:

		Receive, j		
		s_1	s_2	s_3
Send, i	s_1	0.8	0.1	0.1
	s_2	0.05	0.9	0.05
	s_3	0.02	0.08	0.9

The entries list $P(s_j \text{ received} \mid s_i \text{ sent})$. For example, if s_1 is sent, the conditional probability of receiving s_3 is 0.1. (a) Compute $P(s_j \text{ is received})$ for $1 \leq j \leq 3$.
(b) Compute the probabilities $P(s_i \text{ sent} \mid s_j \text{ received})$ for $i, j = 1, 2, 3$.

3. Randomly selected die
There are three dice in a bag. One has one red face, another has two red faces, and the third has three red faces. One of the dice is drawn at random from the bag, each die having an equal chance of being drawn. The selected die is repeatedly rolled.
(a) What is the probability that red comes up on the first roll?
(b) Given that red comes up on the first roll, what is the conditional probability that red comes up on the second roll?
(c) Given that red comes up on the first three rolls, what is the conditional probability that the selected die has red on three faces?

4. Paging a mobile station, with possible misses

A mobile station (MS) is in one of four disjoint cells, numbered 1 through 4. When the MS must be found, it is paged in one cell at at time. Due to channel fading, whenever the MS is paged in the right cell, the page is successful with probability 0.9, and otherwise the page is a miss. The MS is first paged in cell 1. If it is not found there (this happens if the MS wasn't in cell 1, or if it was in cell 1 and the first page was a miss), it is paged in cell 2. If it is not found there, it is paged in cell 3. If it is not found there, it is paged in cell 4. If the MS is not found after all four pages, a second round of pages is started. If the MS hasn't been found after a second round of pages, the overall search is a failure. Let E_i denote the event that the MS is located in cell i, and suppose that $P(E_i) = \frac{5-i}{10}$ for $1 \leq i \leq 4$. Let F_k be the event that the MS is found on the k^{th} page, and let G_k be the event that the MS is not found within the first k pages, for $1 \leq k \leq 8$.

(a) Find $P(G_8)$, which is the probability the overall search is a failure.

(b) Find $P(F_2|G_1)$.

(c) Find $P(F_4|G_3.)$

(d) Find $P(F_8|G_7)$.

5. Let's make a deal

Dilbert is a contestant on Monty Hall's "Let's Make a Deal." Monty shows him 3 curtains. One of the curtains conceals a prize, the other two, junk. Let E_i denote the event that the prize is behind curtain i, for $1 \leq i \leq 3$. The events E_1, E_2, E_3 form a partition of the probability space. We assume that $P(E_i) = \frac{1}{3}$ for each i. Dilbert selects a curtain. For ease of notation, let us suppose without loss of generality that Dilbert initially selects curtain 1.

(a) At this point in the game, as always, Monty, who knows where the prize is, opens one of the remaining curtains to reveal junk. (If both the other curtains cover junk, he selects one at random with equal probability.) He then offers Dilbert the "new improved deal" — Dilbert can either stick to his original guess or he can switch. Let F denote the event that Monty reveals junk behind curtain number 2. (i) Find $P(E_1|F)$, which is the conditional probability, given F, that Dilbert wins if he sticks to his original choice. (ii) Find $P(E_3|F)$, which is the conditional probability Dilbert wins, given F, if he switches. Should he switch?

(b) In a different version this game, Monty calls Wally down from the audience to join Dilbert and asks Dilbert to pick a curtain, and then Wally to pick a different curtain. Monty then opens one of the two curtains *that was picked*, revealing junk behind it and sends that person back to the audience. (Monty always opens one of the two curtains picked by his contestants. If one of the two curtains conceals the prize, he is forced to open the other. If neither curtain conceals the prize, he opens one at random, with each choice having equal probability.) After that, the other player is offered the choice of staying with his original choice or switching to the remaining curtain. For ease of notation, let's assume that Dilbert initially selects curtain 1, and Wally initially selects curtain 2. We view these choices as not being random. Let G be the event that Monty begins by revealing junk behind curtain 2, so that Wally is sent back to the audience. (i) Find $P(E_1|G)$, which is the conditional probability, given G, that Dilbert wins if he sticks to his original choice. (ii) Find $P(E_3|G)$, which is the conditional probability, given G, that Dilbert wins if he switches. Should he switch?

6. Let's make a deal–revisited

In this problem, we analyze the previous problem without using Bayes rule.

(a) Consider the first version of the game, and suppose as before that Dilbert initially selects curtain 1. Suppose Dilbert also made up his mind about what strategy to use before he started to play. What is Dilbert's (absolute) probability of winning the prize (i) if he uses the "stick to original guess" strategy? (ii) if he uses the "switch" strategy? (Hint: On which of the events E_i does he win?) Which strategy is better?

(b) Similarly, consider the second version of the game. Suppose as before that Dilbert selects curtain 1 and Wally selects curtain 2. Compute the probability that Dilbert wins if before he starts play he decides he will (i) use the "stick to original guess" strategy. (ii) Use the "switch" strategy. Which strategy is better?