University of Illinois **Spring 2007**
ECE 413: Problem Set 3 **Due 2/7/07 at beginning of class**

Functions of discrete random variables, and parameter estimation

Assigned reading: *Ross* Sections 4.1-4.6.
Noncredit exercises: Chapter 4, *Ross*, problems 1-42, theoretical exercises 1-9, and self test problems 1-10.

1. Illustration of the law of the unconscious statistician (LOTUS) with a ramp pmf
Let $n \geq 1$ be an integer and suppose the random variable X has the pmf

$$p_X(k) = \begin{cases} \frac{2k}{n(n+1)} & 1 \leq k \leq n \\ 0 & \text{else.} \end{cases}$$

(a) Verify that p_X is a valid pmf.
(b) Let $Y = 1/X$. Describe the pmf of the random variable Y, and sketch it for the case $n = 3$.
(c) Compute $E[Y]$, for general $n \geq 1$, using your answer to part (b).
(d) Compute $E[Y]$ using the law of the unconscious statistician. (You should get the same answer as in part (c). Which way was easier?)

2. Mean, variance, and exponential moments of the Poisson distribution
Let $\lambda \geq 0$. A Poisson random variable with parameter λ is a random variable X with discrete mass function p_X given by $p_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$ for nonnegative integers i. In the degenerate special case that $\lambda = 0$, $p_X(0) = 1$ and $p_X(i) = 0$ for $i \geq 1$.
(a) Show that the mass function sums to one. (Hint: Recall that the power series expansion for e^u, which is absolutely convergent for all u, is given by: $e^u = 1 + u + \frac{u^2}{2!} + \ldots$.)
(b) Find $P(X = 0)$ (i.e. $p_X(0)$) for a Poisson random variable with parameter $\lambda \geq 0$.
(c) Express the mean $E[X]$ in terms of λ.
(d) Express $E[X(X-1)]$ in terms of λ, using LOTUS.
(e) Using the answers to parts (c) and (d), and the fact $E[X(X-1)] = E[X^2] - E[X]$, express $\mathrm{Var}(X)$ in terms of λ.
(f) Express $E[z^X]$ in terms of z and λ, for an arbitrary constant z.

3. Estimation of the parameter of a Poisson random variable
(a) Suppose X is a Poisson random variable with parameter $\lambda \geq 0$, such that the value of λ is unknown, but is to be estimated from observing the value of X. For example, X could represent the number of photons detected over a fixed interval of time by a deep space photon detector pointing at a faint star. Suppose the value of X observed is $X = n$, for some nonnegative integer n. What is the maximum likelihood estimate, $\widehat{\lambda}$, as a function of n?
(b) Suppose Y is a Poisson random variable with parameter $\lambda = b^2$, where b is an unknown parameter with $b \geq 0$, which is to be estimated from observation of Y. Suppose the value of Y observed is $Y = m$, for some nonnegative integer m. What is the maximum likelihood estimate $\widehat{b}$, as a function of m?

4. Parameter estimation for an increasing ramp pmf
Let n be a positive integer, and suppose X has the pmf

$$p_X(k) = \begin{cases} \frac{2k}{n(n+1)} & 1 \le k \le n \\ 0 & \text{else} \end{cases}$$

Suppose n is unknown, but is to be estimated from observation of X. Suppose a particular value k is observed for X. What is the maximum likelihood estimate $\widehat{n}$ of n?

5. Estimating n from observation of a binomial random variable
Suppose a computer wafer of area 100 cm^2 contains n defects, where n is to be estimated. A region of the chip, with one quarter of the area, is carefully examined. Since the region has a quarter of the area of the chip, it is assumed that each defect in the chip falls in the region with probability 0.25. The locations of the defects are assumed to form a sequence of independent trials. Let X be the number of defects found in the region. (Hint: One piece of information in the problem statement was added for realism, but is not used in the solution.) (a) What is the pmf of X?
(b) Given that $X = 8$, what is the maximum likelihood estimate for n, the total number of defects in the chip?