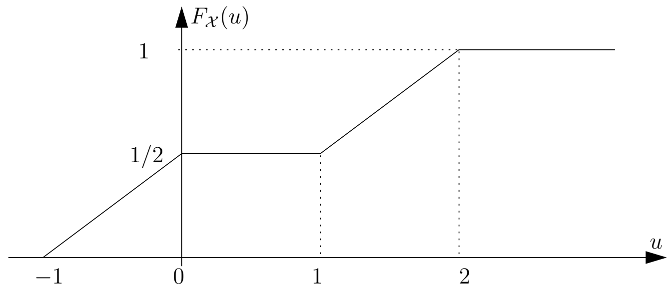
University of Illinois Spring 2007 ECE 413 Final Exam Solutions

1. [25 points] Suppose the CDF of a continuous random variable is as shown below.

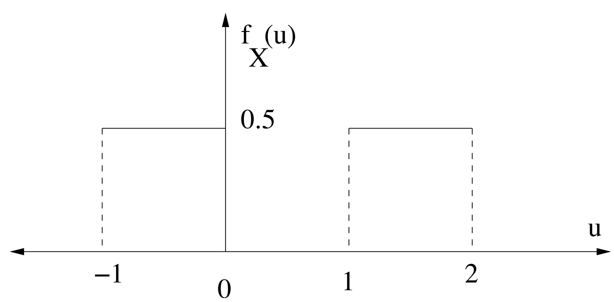


(a) [5 points] Find $P\{|X| \leq 0.8\}$.

(b) [8 points] Find $\mathsf{E}[X]$.

(c) [12 points] Find $\mathsf{var}(X)$.

(a) $P\{|X| \leq 0.8\} = P\{-0.8 \leq X \leq 0.8\} = F_X(0.8) - F_X(-0.8) = \frac{1}{2} - 0.1 = 0.4$.
(b) By the area rule, $E[X]$ is the area of the region to the right of the origin between the CDF and the line $v = 1$, minus the area to the left of the origin, between the CDF and the u axis. By inspection, the areas of the triangular parts cancel, so that $E[X] = 0.5$. For another method, we note that the pdf, the derivative of the CDF, is given by the following sketch:



By inspection, we see that f_X is obtained by starting with an even function, and then shifting it to the right by 0.5. So $E[X] = 0.5$. For yet another method, one can directly use $E[X] = \int_{-\infty}^{\infty} u f_X(u) du$.
(c) Since adding a constant to a random variable, or, equivalently, sliding over the pdf, does not change the variance of the random variable, the variance of X is the same as if the pdf were shifted to the left by 0.5, making it an even function. But then the mean is zero, so we find $\text{Var}(X) = 2\int_{0.5}^{1.5} 0.5u^2 du = \frac{13}{12}$.

2. [25 points] X denotes a continuous random variable with pdf $f_X(u) = \begin{cases} 2u, & 0 \leq u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

(a) [10 points] Find $P\{X \geq 0.4 \,|\, X \leq 0.8\}$.

(b) [15 points] Find the pdf of $Y = -\ln X$. Give both an equation defining the pdf over the whole real line and a sketch of the pdf.

(a) The CDF on the interval $[0, 1]$ is given by $F_X(u) = u^2$. So
$P[X \geq 0.4 | X \leq 0.8] = P[0.4 \leq X \leq 0.8 | X \leq 0.8] = (0.8^2 - 0.4^2)/0.8^2 = \frac{3}{4}$.
(b) The range of Y is the interval $[0, +\infty)$. For $c \geq 0$,
$P[-\log(X) \leq c] = P[\log(X) \geq -c] = P[X \geq e^{-c}] = \int_{e^{-c}}^{1} 2udu = 1 - e^{-2c}$ so $f_Y(c) = \begin{cases} 2\exp(-2c) & c \geq 0 \\ 0 & \text{else} \end{cases}$
That is, Y is an exponential random variable with parameter 2.

3. **[24 points]** A particular webserver may be working or not working. If the webserver is not working, any attempt to access it fails. Even if the webserver is working, an attempt to access it can fail due to network congestion beyond the control of the webserver. Suppose that the *a priori* probability that the server is working is 0.8. Suppose that if the server is working, then each access attempt is successful with probability 0.9, independently of other access attempts. Find the following quantities.

(**a**) **[6 points]** P[first access attempt fails] =

(**b**) **[6 points]** P[server is working | first access attempt fails]=

(**c**) **[6 points]** P[second access attempt fails | first access attempt fails]=

(**d**) **[6 points]** P[server is working | first and second access attempts fail]=

(a) P[first attempt fails]=0.2+(0.8)(0.1)=0.28

(b) P[server is working | first attempt fails] =
P[server working, first attempt fails]/P[first attempt fails] =(0.8)(0.1)/0.28≈ 0.286

(c) P[second attempt fails | first attempt fails] =P[first two attempts fail]/P[first attempt fails] = $[0.2 + (0.8)(0.1)^2]/0.28 \approx 0.783$

(d) P[server is working | first and second attempts fail] =P[server is working and first two attempts fail]/P[first two attempts fail] = $(0.8)(0.1)^2/[0.2 + (0.8)(0.1)^2] \approx 0.0385$

4. **[36 points]** On the basis of a sensor output X, it is to be decided which hypothesis is true: H_0 or H_1. Suppose that if H_0 is true then X has density f_0 and if H_1 is true then X has density f_1, where the densities are given by

$$f_0(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & |u| > 1 \end{cases} \qquad f_1(u) = \begin{cases} |u| & |u| \leq 1 \\ 0 & |u| > 1 \end{cases}$$

(**a**) **[6 points]** Describe the maximum likelihood (ML) decision rule for deciding which hypothesis is true for observation X.

(**b**) **[6 points]** Find the probability of false alarm, p_{false_alarm}, for the ML rule.

(**c**) **[6 points]** Find the probability of missed detection, p_{miss}, for the ML rule.

For the remainder of this problem, use the prior distribution $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

(**d**) **[6 points]** Find the average error probability, p_e, for the ML rule.

(**e**) **[6 points]** Describe the MAP decision rule for deciding which hypothesis is true for observation X.

(**f**) **[6 points]** Find the average error probability, p_e, for the MAP rule.

(a) The likelihood ratio is given by $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = 2|u|$ for $|u| \leq 1$, and the likelihood ratio is not defined for other values of u. (The probability is zero that the observation will be outside the interval $[-1, 1]$ under either hypothesis, anyway.) For the ML rule, the threshold is one, so the ML rule is to decide H_1 if $|X| \geq 0.5$ and decide H_0 otherwise.
(b) This leads to $p_{false\_alarm} = P(|X| \geq 0.5|H_0) = 0.5$.
(c) and $p_{miss} = P(|X| \leq 0.5|H_1) = 0.25$.
(d) In general, $p_e = \pi_0 p_{false\_alarm} + \pi_1 p_{miss}$, which for the ML rule yields $p_e = 0.6 \cdot 0.5 + 0.4 \cdot 0.25 = 0.4$.
(e) The MAP rule is also a likelihood ratio test, but with threshold $\frac{\pi_0}{\pi_1} = 1.5$ applied to the likelihood ratio. So the MAP rule is to decide H_1 if $|X| \geq 0.75$ and decide H_0 otherwise.
(f) For the MAP rule, $p_{false\_alarm} = P(|X| \geq 0.75|H_0) = 0.25$ and $p_{miss} = P(|X| \leq 0.75|H_1) = \frac{9}{16}$, so $p_e = 0.6 \cdot 0.25 + 0.4 \cdot \frac{9}{16} = 0.15 + 0.225 = 0.375$. (Note that, as necessary, the answer to part (f) is less than or equal to the answer to part (d).)

5. **[24 points]** Suppose X and Y are jointly Gaussian with parameters $\mu_X = \mu_Y = 0$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 9$, and $\rho = 0.5$. Let $W = 2X + 3Y$ and $Z = X + aY$, where a is a real valued constant.

   (a) **[6 points]** Express $\text{Cov}(W, Z)$ as a simple function of a.

   (b) **[6 points]** For what value(s) of a are W and Z jointly Gaussian?

   (c) **[6 points]** For what value(s) of a are W and Z independent?

   (d) **[6 points]** For what value(s) of a do W and Z fail to be jointly continuous (i.e., fail to have a joint pdf)?

(a) $\text{Cov}(W, Z) = 2\text{Cov}(X, X) + (3 + 2a)\text{Cov}(X, Y) + 3a\text{Cov}(Y, Y)$
$= 2\sigma_Y^2 + (3 + 2a)\sigma_X\sigma_Y\rho + 3a\sigma_Y^2 = 2 \cdot 4 + (3 + 2a) \cdot 2 \cdot 3 \cdot 0.5 + 3a \cdot 9 = 17 + 33a.$

(b) For all a, because linear transformations preserve the property of being jointly Gaussian.

(c) For $a = -\frac{17}{33}$, because for that value of a, $\text{Cov}(W, Z) = 0$ (i.e. W and Z are uncorrelated) which, since they are jointly Gaussian, makes them independent.

(d) If $a = 1.5$, then $W = 2Z$. That is, the point (W, Z) falls on the line $\{(w, z) : w = 2z\}$ with probability one. Thus, there is no joint pdf for that case.

6. **[56 points]** The jointly continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(u, v) = \begin{cases} 2/3, & 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le u + v \le 1, \\ 4/3, & 0 \le u \le 1,\ 0 \le v \le 1,\ 1 < u + v \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) **[8 points]** Find the marginal pdf of X and **draw a neat sketch** of $f_X(u)$.

   (b) **[8 points]** Find the conditional pdf of Y given that $X = 2/3$ and **draw a neat sketch** of $f_{Y\,|\,X}(v\,|\,2/3)$.

   (c) **[8 points]** Find $P\{X^2 + Y^2 \le 1\}$.

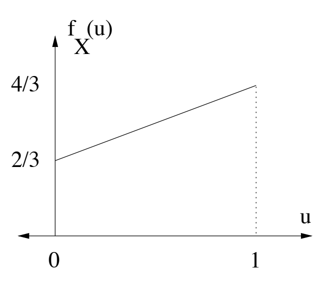
   (d) **[8 points]** Find $P\{X^2 + Y^2 \le 1\,|\,(X - 1)^2 + (Y - 1)^2 \le 1\}$.

   (e) **[8 points]** Find $P\{X + Y \le \alpha\}$ where α is some fixed number in the range $0 \le \alpha < 1$.

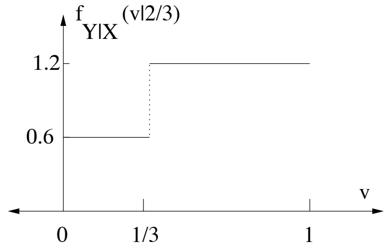
   (f) **[8 points]** Find $P\{X + Y \le \alpha\}$ where α is some fixed number in the range $1 \le \alpha < 2$.

   (g) **[8 points]** From your answers to parts (d) and (e), find the pdf of the random variable $\mathcal{Z} = X + Y$.

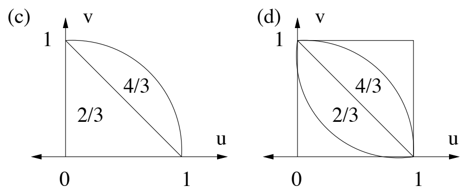
(a) The support of f_X is the interval $[0, 1]$, so let $0 \le u \le 1$. Then $f_X(u) = \int_0^{1-u} \frac{2}{3} dv + \int_{1-u}^{1} \frac{4}{3} dv = \frac{2(1-u)+4u}{3} = \frac{2(1+u)}{3}$. That is, $f_X(u) = \begin{cases} \frac{2(1+u)}{3}, & 0 \le u \le 1 \\ 0, & \text{else} \end{cases}$



(b) $f_X(2/3) = \frac{10}{9}$ by part (a). Dividing this into the joint density yields $f_{Y|X}(v|2/3) = \begin{cases} 0.6 & 0 \le v \le \frac{1}{3} \\ 1.2 & \frac{1}{3} < v \le 1 \\ 0 & \text{else} \end{cases}$



(c) The region is the union of the triangle of area 0.5, over which the density is 2/3, and a cap of a circle, of area $\frac{\pi}{4} - 0.5$, over which the density is 4/3. Thus, $P\{X^2 + Y^2 \leq 1\} = (0.5)\frac{2}{3} + (\frac{\pi}{4} - 0.5)\frac{4}{3} = \frac{\pi - 1}{3}$.

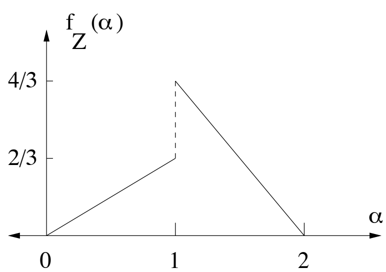


(d) Similarly, $P\{(X-1)^2 + (Y-1)^2 \leq 1\} = (0.5)\frac{4}{3} + (\frac{\pi}{4} - 0.5)\frac{2}{3} = \frac{\pi+2}{6}$, and the relevant intersection (the union of the two caps shown in the figure) has probability $(\frac{2}{3} + \frac{4}{3})(\frac{\pi}{4} - 0.5) = \frac{\pi - 2}{2}$. Hence $P\{X^2 + Y^2 \leq 1 \mid (X-1)^2 + (Y-1)^2 \leq 1\} = \frac{\pi-2}{2} / \frac{\pi+2}{6} = \frac{3(\pi-2)}{\pi+2}$.

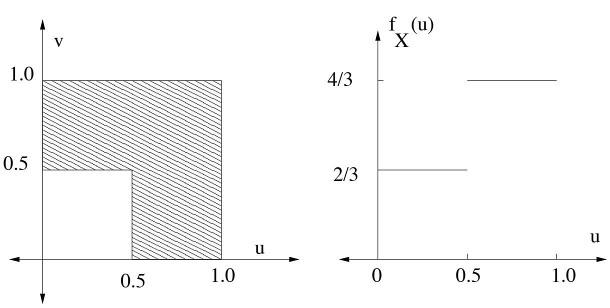
(e) This is given by $\frac{2}{3} \cdot \frac{1}{2} \cdot \alpha^2 = \frac{\alpha^2}{3}$.

(f) Simlarly, this is given by $1 - \frac{4}{3} \cdot \frac{1}{2} \cdot (2-\alpha)^2 = 1 - \frac{2(2-\alpha)^2}{3}$.

(g) Thus, $f_Z(\alpha) = \begin{cases} \frac{2\alpha}{3} & 0 \leq \alpha \leq 1 \\ \frac{4(2-\alpha)}{3} & 1 \leq \alpha \leq 2 \\ 0 & \text{else} \end{cases}$



7. **[35 points]** Let (X, Y) be uniformly distributed over the region shown. Also shown, for your convenience, is the pdf of X, which is the same as the pdf of Y.



(**a**) **[7 points]** Find the function $g(X)$ which gives the minimum mean square error (MMSE) estimator of Y given X.

(b) **[7 points]** Find the (average) mean square error for the MMSE estimator of Y given X.

(c) **[7 points]** Find $E[X]$.

(d) **[7 points]** Find $E[XY]$.

(e) **[7 points]** *Sketch* the linear function $L(u)$, of the form $au+b$, such that $L(X)$ is the linear estimator of Y based on X with the minimum MSE. (You are *not* required to compute the exact numerical values, but the smaller the MSE for the linear estimator you sketch, the higher the score.)

(a) Note that if $0 \le u \le 0.5$, the conditional distribution of Y given $X=u$ is the uniform distribution over the interval $[0.5, 1.0]$, which has mean 0.75 and variance $\frac{(0.5)^2}{12} = \frac{1}{48}$. Whereas if $0.5 < u \le 1.0$, the conditional distribution of Y given $X=u$ is the uniform distribution over the interval $[0,1]$, which has mean 0.50 and variance $\frac{1}{12}$. Therefore, $g(u) = E(Y|X=u) = \begin{cases} 0.75 & 0 \le u \le 0.5 \\ 0.5 & 0.5 < u \le 1.0 \end{cases}$. (The value of $g(u)$ for $u<0$ or $u>1$ is arbitrary.)

(b) By the reasoning above, the conditional MSE is $\frac{1}{48}$ if $0 \le u \le 0.5$ and is $\frac{1}{12}$ if $0.5 < u \le 1.0$. To find the average MSE we integrate the conditional MSE times f_X, which has 1/3 of its mass on the interval $[0, 0.5]$ and 2/3 on the interval $[0.5, 1]$, yielding MSE$=\frac{1}{48}\cdot\frac{1}{3}+\frac{1}{12}\cdot\frac{2}{3} = \frac{1+8}{144} = \frac{1}{16} = 0.0625$.

(c) $E[X] = \int_0^{0.5} \frac{2}{3} u\,du + \int_{0.5}^1 \frac{4}{3} u\,du = \frac{1}{12} + \frac{1}{2} = \frac{7}{12}$.

(d) Letting S denote the support of f_{XY}, we have

$$\begin{aligned} E[XY] = \int\int_S uv\frac{4}{3}dudv &= \frac{4}{3}\left\{\int_0^1\int_0^1 uvdudv - \int_0^{0.5}\int_0^{0.5} uvdudv\right\} \\ &= \frac{4}{3}\left\{\int_0^1 udu\int_0^1 vdv - \int_0^{0.5} udu\int_0^{0.5} vdv\right\} \\ &= \frac{4}{3}\left\{\frac{1}{2^2} - \frac{1}{8^2}\right\} = \frac{4}{3}\cdot\frac{15}{64} = \frac{5}{16} \end{aligned}$$

(Note: It can also be shown that $\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{11}{144}$, $\mathrm{Cov}(X,Y) = -\frac{1}{36}$, $\rho = -\frac{4}{11}$, $L(u) = \frac{7}{12} - \frac{4}{11}(u - \frac{7}{12})$, and the MSE for L is $\frac{35}{528} \approx 0.0663$, only 6% larger than the MSE for the best nonlinear estimator. Also, $L(0) \approx 0.801$ and $L(1) \approx 0.435$.)

(e) The figure below shows $L(u)$, as well as the optimal nonlinear estimator (dashed lines).