University of Illinois Spring 2007

ECE 413: Final Examination

Tuesday May 8, 2007
1:30 p.m. — 4:30 p.m.
269 Everitt Laboratory

Name: ____________________________________

Section: ☐ C, 10 am ☐ D, 11 am

University ID Number: ______________________

Signature: ___________________________________

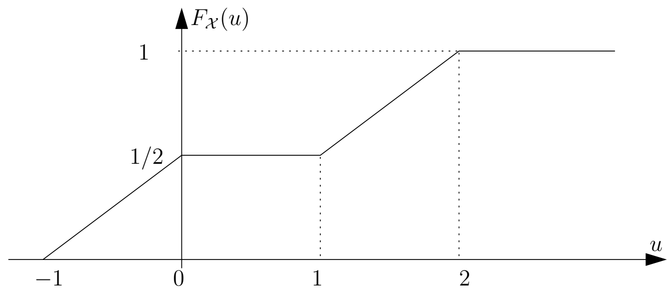
Instructions

This exam is closed book and closed notes except that *three* 8.5"×11" sheets of notes are permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.
The exam consists of 7 problems worth a total of 225 points. Note that the problems are not weighted equally and pace yourself accordingly. Write your answers in the spaces provided.
SHOW YOUR WORK. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page.

Grading

1. 25 points	___________
2. 25 points	___________
3. 24 points	___________
4. 36 points	___________
5. 24 points	___________
6. 56 points	___________
7. 35 points	___________
Total (225 points)	___________

1. [**25 points**] Suppose the CDF of a continuous random variable is as shown below.



(**a**) [**5 points**] Find $P\{|X| \leq 0.8\}$.

(**b**) [**8 points**] Find $\mathsf{E}[X]$.

(**c**) [**12 points**] Find $\mathsf{var}(X)$.

2. [**25 points**] X denotes a continuous random variable with pdf $f_X(u) = \begin{cases} 2u, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

(**a**) [**10 points**] Find $P\{X \geq 0.4 \,|\, X \leq 0.8\}$.

(**b**) [**15 points**] Find the pdf of $Y = -\ln X$. Give both an equation defining the pdf over the whole real line and a sketch of the pdf.

3. **[24 points]** A particular webserver may be working or not working. If the webserver is not working, any attempt to access it fails. Even if the webserver is working, an attempt to access it can fail due to network congestion beyond the control of the webserver. Suppose that the *a priori* probability that the server is working is 0.8. Suppose that if the server is working, then each access attempt is successful with probability 0.9, independently of other access attempts.

(**a**) **[6 points]** Find $P[$ first access attempt fails$]$.

(**b**) **[6 points]** Find $P[$server is working | first access attempt fails $]$.

(**c**) **[6 points]** Find $P[$second access attempt fails | first access attempt fails $]$.

(**d**) **[6 points]** Find $P[$server is working | first and second access attempts fail $]$.

4. **[36 points]** On the basis of a sensor output X, it is to be decided which hypothesis is true: H_0 or H_1. Suppose that if H_0 is true then X has density f_0 and if H_1 is true then X has density f_1, where the densities are given by

$$f_0(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & |u| > 1 \end{cases} \qquad f_1(u) = \begin{cases} |u| & |u| \leq 1 \\ 0 & |u| > 1 \end{cases}$$

(**a**) **[6 points]** Describe the maximum likelihood (ML) decision rule for deciding which hypothesis is true for observation X.

(**b**) **[6 points]** Find the probability of false alarm, p_{false_alarm}, for the ML rule.

(**c**) **[6 points]** Find the probability of missed detection, p_{miss}, for the ML rule.

For the remainder of this problem, use the same densities:

$$f_0(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & |u| > 1 \end{cases} \qquad f_1(u) = \begin{cases} |u| & |u| \leq 1 \\ 0 & |u| > 1 \end{cases}$$

and use the prior distribution $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

(**d**) [**6 points**] Find the average error probability, p_e, for the ML rule.

(**e**) [**6 points**] Describe the MAP decision rule for deciding which hypothesis is true for observation X.

(**f**) [**6 points**] Find the average error probability, p_e, for the MAP rule.

5. **[24 points]** Suppose X and Y are jointly Gaussian with parameters $\mu_X = \mu_Y = 0$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 9$, and $\rho = 0.5$. Let $W = 2X + 3Y$ and $Z = X + aY$, where a is a real valued constant.

(**a**) **[6 points]** Express $\text{Cov}(W, Z)$ as a simple function of a.

(**b**) **[6 points]** For what value(s) of a are W and Z jointly Gaussian?

(**c**) **[6 points]** For what value(s) of a are W and Z independent?

(**d**) **[6 points]** For what value(s) of a do W and Z fail to be jointly continuous (i.e., fail to have a joint pdf)?

6. [**56 points**] The jointly continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} 2/3, & 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le u+v \le 1, \\ 4/3, & 0 \le u \le 1,\ 0 \le v \le 1,\ 1 < u+v \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(**a**) [**8 points**] Find the marginal pdf of X and **draw a neat sketch** of $f_X(u)$.

(**b**) [**8 points**] Find the conditional pdf of Y given that $X = 2/3$ and **draw a neat sketch** of $f_{Y\,|\,X}(v\,|\,2/3)$.

(**c**) [**8 points**] Find $P\{X^2 + Y^2 \le 1\}$.

(**d**) [**8 points**] Find $P\{X^2 + Y^2 \le 1\,|\,(X-1)^2 + (Y-1)^2 \le 1\}$.

The remainder of this problem uses the same pdf:

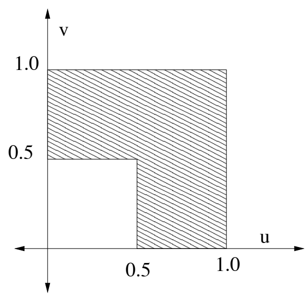
$$f_{X,Y}(u,v) = \begin{cases} 2/3, & 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le u+v \le 1, \\ 4/3, & 0 \le u \le 1,\ 0 \le v \le 1,\ 1 < u+v \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

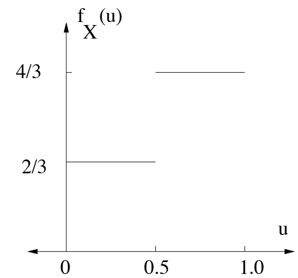
(**e**) [**8 points**] Find $P\{X+Y \le \alpha\}$ where α is some fixed number in the range $0 \le \alpha < 1$.

(**f**) [**8 points**] Find $P\{X+Y \le \alpha\}$ where α is some fixed number in the range $1 \le \alpha < 2$.

(**g**) [**8 points**] From your answers to parts (d) and (e), find the pdf of the random variable $\mathcal{Z} = X+Y$.

7. [**35 points**] Let (X, Y) be uniformly distributed over the region shown. Also shown, for your convenience, is the pdf of X, which is the same as the pdf of Y.

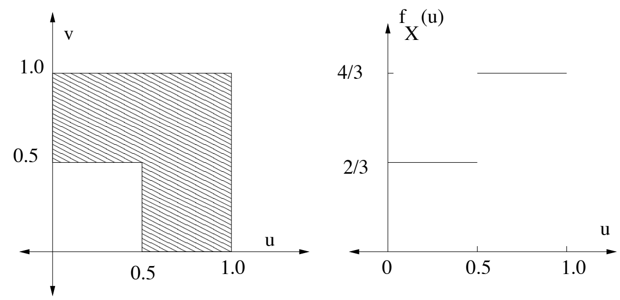




(**a**) [**7 points**] Find the function $g(X)$ which gives the minimum mean square error (MMSE) estimator of Y given X.

(**b**) [**7 points**] Find the (average) mean square error for the MMSE estimator of Y given X.

The remainder of this problem uses the same pdfs:



(**c**) [**7 points**] Find $E[X]$.

(**d**) [**7 points**] Find $E[XY]$.

(**e**) [**7 points**] *Sketch* the linear function $L(u)$, of the form $au + b$, such that $L(X)$ is the linear estimator of Y based on X with the minimum MSE. (You are *not* required to compute the exact numerical values, but the smaller the MSE for the linear estimator you sketch, the higher the score.)