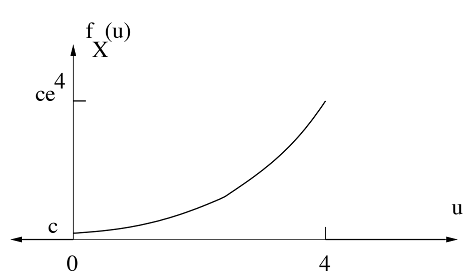
University of Illinois, ECE 413 Spring 2007 Solutions to exam 2

1. [**24 points**] A random variable X is observed. Under hypothesis H_0, X has the Poisson distribution with mean $\lambda_0 = 1$. Under hypothesis H_1, X has the Poisson distribution with mean $\lambda_1 = 3$. **(a)** [**8 points**] Describe the maximum likelihood decision rule for selecting one of the hypotheses, given that $X = n$ is observed. Be as explicit as possible. (Hint: $e \approx 2.7$, $e^2 \approx 7.4$, $e^3 \approx 20$, $e^4 \approx 54$, $e^5 \approx 148$.) **(b)** [**8 points**] For what values of the prior probability π_1 (if any) does the MAP rule select hypothesis H_1 for all $n \geq 0$? **(c)** [**8 points**] Suppose that two observations are available, instead of one. Suppose these observations are made under the same hypothesis, and that, given which hypothesis is true, the two observations are conditionally independent, and each has the same distribution as X above. Let n_1 and n_2 denote the two observations. Finally, suppose the prior probabilities assigned to the hypotheses are $\pi_0 = 0.8$ and $\pi_1 = 0.2$. Identify all pairs (n_1, n_2) such that the MAP rule decides that H_1 is true. Be as explicit as possible.

SOLUTION (a) The likelihood ratio is $\Lambda(n) = \frac{e^{-3}3^n/n!}{e^{-1}1^n/n!} = \frac{3^n}{e^2} \approx \frac{3^n}{7.4}$. Hence, for a nonnegative integer n, $\Lambda(n) > 1$ if and only if $n \geq 2$. Thus, the ML rule is to decide H_0 is true if and only if $n \geq 2$.
(b) That is, for what values of (π_0, π_1) is $\frac{3^n}{e^2} \geq \frac{\pi_0}{\pi_1}$ for all $n \geq 0$? The likelihood ratio is minimized by $n = 0$, and the minimum value is $\frac{1}{e^2}$. So if $\frac{1}{e^2} > \frac{\pi_0}{\pi_1}$, or equivalently, $\pi_0 < \frac{1}{1+e^2} \approx \frac{1}{8.4}$, then the MAP rule will always declare H_1.
(c) The likelihood ratio is $\frac{(e^{-3}3^{n_1}/n_1!)(e^{-3}3^{n_2}/n_2!)}{(e^{-1}1^{n_1}/n_1!)(e^{-1}1^{n_2}/n_2!)}, = \frac{3^{n_1+n_2}}{e^4} \approx \frac{3^{n_1+n_2}}{54}$ and the MAP threshold is $\frac{\pi_0}{\pi_1} = 4$. Therefore, the map rule is to decide H_1 if and only if $\frac{3^{n_1+n_2}}{54} > 4$ or, equivalently, if and only if $n_1 + n_2 \geq 5$.

2. [**20 points**] Let X be a random variable with pdf $f_X(u) = \begin{cases} ce^u & \text{if } 0 \leq u \leq 4 \\ 0 & \text{else.} \end{cases}$ For all parts of this problem, you can use terms such as e^4 in your answer. Numerical values are not required. **(a)** [**2 points**] Sketch f_X. **(b)** [**6 points**] Find the constant c. **(c)** [**6 points**] Find $P(X \geq 3 \mid 2 \leq X \leq 6)$. **(d)** [**6 points**] Find $P(2 \leq X \leq 6 \mid X \geq 3)$.

SOLUTION (a)



(b) $1 = \int_0^4 ce^u du = c(e^4 - 1)$ so that $c = \frac{1}{e^4-1}$.
(c) $P(\{X \geq 3\} \cap \{2 \leq X \leq 6\}) = P(3 \leq X \leq 6) = P(3 \leq X \leq 4) = \int_3^4 ce^u du = c(e^4 - e^3)$ and $P(2 \leq X \leq 6) = P(2 \leq X \leq 4) = \int_2^4 ce^u du = c(e^4 - e^2)$.
So, $P(X \geq 3 \mid 2 \leq X \leq 6) = \frac{e^4-e^3}{e^4-e^2} = \frac{e^2-e}{e^2-1} = \frac{e(e-1)}{(e+1)(e-1)} = \frac{e}{e+1}$.
(d) $P(\{2 \leq X \leq 6\} \cap \{X \geq 3\}) = P(3 \leq X \leq 6) = P(3 \leq X \leq 4)$ and $P(X \geq 3) = P(3 \leq X \leq 4)$.
So, $P(2 \leq X \leq 6 \mid X \geq 3) = 1$.

3. **[20 points]** A professor breaks the chalk piece with which he is writing on the blackboard at random times that can be modeled as arrivals in a Poisson process with arrival rate $\lambda = 0.2$ per minute. **(a)** **[6 points]** What is the expected length of time between two successive chalk breaks? **(b)** **[6 points]** What is the average number of times that the professor breaks the chalk during a 50 minute lecture? **(c)** **[8 points]** Given that the professor broke 8 chalk pieces in a 50 minute lecture, what is the conditional probability that the professor broke (exactly) 3 chalk pieces in the first 30 minutes?

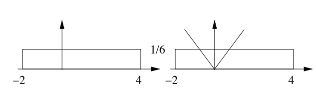
SOLUTION (a) The interarrival time in a Poisson process with arrival rate λ (time between two successive chalk breaks on this instance) is an exponential random variable with parameter λ. Hence, the average length of time between successive chalk-breaks is the mean of this exponential random variable, which is $\lambda^{-1} = 5$ minutes.
(b) The number of times that the professor breaks the chalk during a 50 minute lecture is a Poisson random variable $\mathcal{N}(0, 50]$ with parameter $\lambda \times 50 = 10$ and mean value $E[\mathcal{N}(0, 50]] = 10$.
(c) Given the total number is 8, the times of the breaks can be considered to be independent and uniform over the 50 minute period. Thus, each of the 8 is broken in the first 30 minutes with probability 30/50=0.6, and the conditional distribution of the number broken in the first 30 minutes is binomial with parameters $n = 8$ and $p = 30/50 = 0.6$. So $P(\mathcal{N}(0, 30] = 3|\mathcal{N}(0, 50] = 8) = \binom{8}{3}(0.6)^3(0.4)^5$.

4. **[16 points]** Suppose X is a random variable with mean 10 and variance 3. Find the numerical value of $P\{X < 10 - \sqrt{3}\}$ (or, nearly equivalently, $P\{X < 8.27\}$) for the following two choices of distribution type: **(a)** **[8 points]** Assuming X is a Gaussian random variable. **(b)** **[8 points]** Assuming X is a uniform random variable.

SOLUTION (a) If X is Gaussian, $P\{X < 10 - \sqrt{3}\} = P\{\frac{X-10}{\sqrt{3}} \le -1\} = \Phi(-1) = 1 - \Phi(1) \approx 1 - 0.8413 = 0.1587$.
(b)A random variable uniformly distributed on $[a, b]$ has mean $\dfrac{a+b}{2}$ and variance $\dfrac{(b-a)^2}{12}$. Hence, we have that $b - a = 6$, and $a + b = 20$, giving $a = 7, b = 13$. That is, X is uniformly distributed over $[7, 13]$. Therefore, $P\{X < 8.27\} = \frac{8.27-7}{6} \approx 0.211$.

5. **[20 points]** Suppose Z is a *uniform* random variable on the interval $[-2, 4]$. **(a)** **[10 points]** Find $\mathsf{E}[|Z|]$. **(b)** **[10 points]** Give the pdf of $Y = |Z|$ in equation form (specify it for all v, $-\infty < v < \infty$) *and* sketch the pdf.

SOLUTION (a) The pdf is shown.



$$\mathsf{E}[|Z|] = \frac{1}{6}\left[\int_{-2}^{0} -u\,du + \int_{0}^{4} u\,du\right] = \frac{1}{6}\left[\frac{-u^2}{2}\bigg|_{-2}^{0} + \frac{u^2}{2}\bigg|_{0}^{4}\right] = \frac{1}{6}[2+8] = \frac{5}{3}.$$

(b) $Y = |Z|$ takes on values in $[0, 4]$, and hence $F_Y(v) = 0$ for $v < 0$ and $F_Y(v) = 1$ for $v > 4$.
If $0 \le v \le 2, F_Y(v) = P\{Y \le v\} = P\{-v \le Z \le v\} = v/3 \Rightarrow f_Y(v) = 1/3$.
If $2 \le v \le 4, F_Y(v) = P\{Y \le v\} = P\{Z \le v\} = (v+2)/6 \Rightarrow f_Y(v) = 1/6$.
Else, $v < 0$ or $v > 4$, and $f_Y(v) = 0$.