University of Illinois at Urbana-Champaign

ECE 413: Probability with Engineering Applications

Spring 2007
Exam II

Monday, April 9, 2007

Name: ____

- You have 60 minutes for this exam. The exam is closed book and closed note, except you may consult both sides of two $8.5'' \times 11''$ sheets of notes in ten point font size or larger, or equivalent handwriting size.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (24 pts.)

2. __________ (20 pts.)

3. __________ (20 pts.)

4. __________ (16 pts.)

5. __________ (20 pts.)

Total: ________(100 pts.)

1. **[24 points]** A random variable X is observed. Under hypothesis H_0, X has the Poisson distribution with mean $\lambda_0 = 1$. Under hypothesis H_1, X has the Poisson distribution with mean $\lambda_1 = 3$.

(a) **[8 points]** Describe the maximum likelihood decision rule for selecting one of the hypotheses, given that $X = n$ is observed. Be as explicit as possible. (Hint: $e \approx 2.7,\ e^2 \approx 7.4,\ e^3 \approx 20,\ e^4 \approx 54,\ e^5 \approx 148$.)

(b) **[8 points]** For what values of the prior probability π_1 (if any) does the MAP rule select hypothesis H_1 for all $n \geq 0$?

(c) **[8 points]** Suppose that two observations are available, instead of one. Suppose these observations are made under the same hypothesis, and that, given which hypothesis is true, the two observations are conditionally independent, and each has the same distribution as X above. Let n_1 and n_2 denote the two observations. Finally, suppose the prior probabilities assigned to the hypotheses are $\pi_0 = 0.8$ and $\pi_1 = 0.2$. Identify all pairs (n_1, n_2) such that the MAP rule decides that H_1 is true. Be as explicit as possible.

2. [20 points] Let X be a random variable with pdf $f_X(u) = \begin{cases} ce^u & \text{if } 0 \le u \le 4 \\ 0 & \text{else.} \end{cases}$ For all parts of this problem, you can use terms such as e^4 in your answer. Numerical values are not required.

(a) [2 points] Sketch f_X.

(b) [6 points] Find the constant c.

(c) [6 points] Find $P(X \ge 3 \mid 2 \le X \le 6)$.

(d) [6 points] Find $P(2 \le X \le 6 \mid X \ge 3)$.

3. [20 points] A professor breaks the chalk piece with which he is writing on the blackboard at random times that can be modeled as arrivals in a Poisson process with arrival rate $\lambda = 0.2$ per minute.

(a) [6 points] What is the expected length of time between two successive chalk breaks?

(b) [6 points] What is the average number of times that the professor breaks the chalk during a 50 minute lecture?

(c) [8 points] Given that the professor broke 8 chalk pieces in a 50 minute lecture, what is the conditional probability that the professor broke (exactly) 3 chalk pieces in the first 30 minutes?

4. [16 points] Suppose X is a random variable with mean 10 and variance 3. Find the numerical value of $P\{X < 10 - \sqrt{3}\}$ (or, nearly equivalently, $P\{X < 8.27\}$) for the following two choices of distribution type:

(a) [8 points] Assuming X is a Gaussian random variable.

(b) [8 points] Assuming X is a uniform random variable.

5. [20 points] Suppose Z is a *uniform* random variable on the interval $[-2, 4]$.
(a) [10 points] Find $\mathsf{E}[|Z|]$.

(b) [10 points] Give the pdf of $Y = |Z|$ in equation form (specify it for all v, $-\infty < v < \infty$) *and* sketch the pdf.