University of Illinois Spring 2005

ECE 413: Problem Set 13

Due:	Wednesday April 27 at the beginning of class.
Reading:	Ross, Chapter 6 and 7
Noncredit exercises:	Ross, Chapter 6, Problems 26, 28-30, 41-43, 51, 54;
	Theoretical Exercises: 8, 14, 22, 23, 33;
	Ross Chapter 7: Problems 1, 16, 26, 29, 34, 36;
	Theoretical Exercises: 1, 2, 17, 22, 23, 40

This Problem Set contains five problems

1. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown in the left hand figure below.

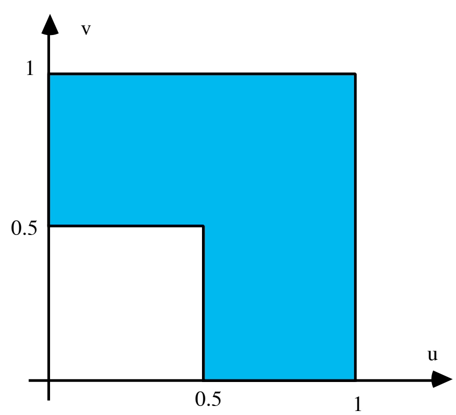
(**a**) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$.

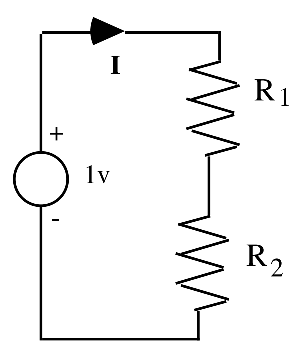
(**b**) *Write down* the marginal pdf $f_{\mathcal{Y}}(v)$ of the random variable $\mathcal{Y}$ from your answer to part (a).

(**c**) Find $P\{\mathcal{X} < \mathcal{Y} < 2\mathcal{X}\}$.

(**d**) Find $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha)$, the conditional pdf of $\mathcal{X}$ given that $\mathcal{Y} = \alpha$, where $0 < \alpha < 1/2$. Find $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha)$, the conditional pdf of $\mathcal{X}$ given that $\mathcal{Y} = \alpha$, where $1/2 < \alpha < 1$.

(**e**) Now, apply the theorem of total probability to compute $f_{\mathcal{X}}(u)$, the *unconditional* pdf of X from $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha)$. Do you get the same answer as in part (a)? Why not?





2. Two resistors are connected in series to a one-volt voltage source as shown in the righthand diagram above. Suppose that the resistance values $\mathcal{R}_1$ and $\mathcal{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval $(0, 1)$. Find the pdf $f_{\mathcal{I}}(a)$ of the current $\mathcal{I}$ (measured in amperes) in the circuit.

3. The number of hours $\mathcal{R}$ that a student spends **r**eading about probability in preparation for the ECE 413 Final Examination, and the number of hours $\mathcal{S}$ that the student spends **s**leeping can be modeled as random variables with joint probability density function

$$f_{\mathcal{R},\mathcal{S}}(x,y) = \begin{cases} K, & x \geq 0, y \geq 0, , 10 \leq x+y \leq 20, \\ 0, & \text{otherwise.} \end{cases}$$

(**a**) What is the value of K?

(**b**) What is the marginal pdf of $\mathcal{R}$?

(**c**) Unfortunately, the more the student tries to read about probability, the more confused the student gets. Also, the less the student sleeps, the more tired the student gets. As a result, the student's percentage score $\mathcal{T}$ on the ECE 413 Final Exam is related to $\mathcal{S}$ and $\mathcal{R}$ via the equation

$$\mathcal{T} = 50 + 2.5(\mathcal{S} - \mathcal{R}).$$

Find the pdf of $\mathcal{T}$.

(**d**) **Noncredit exercise:** Should $\mathcal{S}$ have denoted **s**tudying and $\mathcal{R}$ denoted **r**esting instead?

4. $\mathcal{X}$ and $\mathcal{Y}$ denote *independent* standard Gaussian random variables.

(**a**) What is the joint pdf $f_{\mathcal{X},\mathcal{Y}}(u,v)$ of $\mathcal{X}$ and $\mathcal{Y}$?

(**b**) Sketch the u-v plane and indicate on it the region over which you need to integrate the joint pdf in order to find $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$. [Hint: read the Solutions to Problems 4(b) of Problem Set 1, and Problem 2(c) of Problem Set 2.]
Compute $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$.

(**c**) Let $\mathcal{Z} = \mathcal{X}^2 + \mathcal{Y}^2$. What is the pdf of $\mathcal{Z}$?

(**d**) Express $P\{|\mathcal{X}| > \alpha\}$ in terms of the complementary unit Gaussian CDF function $Q(x)$ (cf. Problem 2 of Problem Set 10), and use this to write $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$ in terms of $Q(x)$. (Remember commas mean intersections).

(**e**) On your sketch of part (b), show the region over which you must integrate the joint pdf to find $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$. Use your sketch to prove the following result: $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$ for $\alpha > 0$.

(**f**) Show that inequality of part (e) implies that $Q(x) < \frac{1}{2}\exp(-x^2/2)$ for $x > 0$ as was proved earlier in Problem 2(b) of Problem Set 10.

(**g**) On your sketch of parts (b) and (d), show the region over which you must integrate to find $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\}$, and prove that

$$P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq \alpha^2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 < 2\alpha^2\}.$$

Use these inequalities to deduce the *lower* bound $Q(x) > \frac{1}{4}\exp(-x^2)$ for $x > 0$. Note that at $x = 0$, equality holds in the upper bound of part (f) but not in this lower bound .

5. The joint pdf of $\mathcal{X}$ and $\mathcal{Y}$ is given by $f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 2u, & 0 < u < 1, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

Find the pdf of $\mathcal{Z} = \mathcal{X}^2\mathcal{Y}$.