University of Illinois Spring 2005

ECE 413: Problem Set 11

Due:	Wednesday April 13 at the beginning of class.
Reminder:	Hour Exam II is on Monday April 11, 7 p.m. in 269 Everitt Lab.
Reading:Reading:	Ross, Chapters 5 and 6
Noncredit exercises:	Ross, Chapter 5, Problems 10-19; 21, 22, 24, 31-41

This Problem Set contains six problems

1. A signal $x(t) = \exp(-\pi t^2), -\infty < t < \infty$, is the input to an ideal low-pass filter whose transfer function is $H(f) = \text{rect}(f/2)$. Let $y(t)$ denote the output of the filter. Find the *numerical* value of $y(0)$. [Hint: $X(f) = \exp(-\pi f^2), -\infty < f < \infty$.]

2. [Read Example 3d (pp. 198-199) in Chapter 5 of Ross first] Let the straight line segment ACB be a diameter of a circle of unit radius and center C. Consider an *arc* AD of the circle where the length $\mathcal{X}$ of the arc (measured clockwise around the circle) is a random variable uniformly distributed on $[0, 2\pi)$. Now consider the *random chord* AD whose length we denote by $\mathcal{L}$.

(**a**) Find the probability that $\mathcal{L}$ is greater than the side of the equilateral triangle inscribed in the circle.

(**b**) Express $\mathcal{L}$ as a function of the random variable $\mathcal{X}$, and find the pdf for $\mathcal{L}$.

3. Ross, p. 231: Problem 36, Chapter 5.

4. Let $\mathcal{X}$ denote the time of the first arrival after $t = 0$ in a Poisson process with arrival rate λ.

(**a**) What is the value of the CDF of $\mathcal{X}$ at time T? that is, what is $P\{\mathcal{X} \leq T\}$?

(**b**) Let A denote the event that there is exactly one arrival in the interval $(0, T]$. What is $P(A)$?

(**c**) Is the $P(A)$ that you found for part (b) the same as the value of $P\{\mathcal{X} \leq T\}$ that you gave in part (a) ? Explain why the two are the same (or are different, as appropriate).

(**d**) For $0 < t < T$, what is the conditional probability that $\{X \leq t\}$ given the event A, that is, given that there was exactly one arrival in $(0, T]$?

5. Consider a Poisson process with arrival rate λ.

(**a**) What is the mean number of arrivals in the interval $(0, 4]$? That is, what is $\mathsf{E}[N(0, 4]]$?

(**b**) What is $P[\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}]$?

(**c**) If we observe that there were 5 arrivals in $(0, 6]$, what is the maximum-likelihood estimate of the arrival rate λ?

(**d**) Now suppose that $\lambda = \ln 2$. What is the probability that at least one arrival occurs in $(0, t]$?

6. If hypothesis H_0 is true, the pdf of $\mathcal{X}$ is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is exponential with parameter 10.

(**a**) Sketch the two pdfs.

(**b**) State the *maximum-likelihood* decision rule in terms of a threshold test on the *observed value* u of the random variable $\mathcal{X}$ instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ to 1.

(**c**) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part(b)?

(**d**) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_0/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable $\mathcal{X}$.

(**e**) If $\pi_0 = 1/3$, what is the *average* probability of error of the Bayesian decision rule?

(**f**) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by $\mathcal{X}$?

(**g**) Show that if $\pi_0 > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by $\mathcal{X}$. What is the average probability of error for the maximum-likelihood rule when $\pi_0 > 2/3$?