University of Illinois Spring 2005

ECE 413: Problem Set 10

Due: Wednesday April 6 at the beginning of class.
Reading: Ross, Chapter 5
Noncredit exercises: Ross, Chapter 5, Problems 10-19; 21, 22, 24, 31-41

This Problem Set contains seven problems

1. As discussed in class, the probability of failure of a TMR system with (perfect majority gate) is $3p^2 - 2p^3$ where p is the probability of failure of each module, and the modules fail independently of each other. Now, suppose that the system is put into operation at $t = 0$, and let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ denote the time of failure of each module. The independence of failures enters into our calculations as the assertion that for all $t_1, t_2, t_3 > 0$, the events $\{\mathcal{X}_1 > t_1\}, \{\mathcal{X}_2 > t_2\}, \{\mathcal{X}_1 > t_3\}$ are independent events. Note that the occurrences of these events are equivalent to the assertions that modules 1, 2, 3 respectively have *not* failed (i.e., are operational) at times t_1, t_2, t_3. We model $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ as exponential random variables with parameter λ.

Let $\mathcal{Y}$ denote the time of failure of the TMR system so that the occurrence of the event $\{\mathcal{Y} > T\}$ means that the TMR system is operational at time T.

(**a**) Express the event $\{Y > T\}$ in terms of unions, intersections and complements of the events $\{\mathcal{X}_1 > T\}, \{\mathcal{X}_2 > T\}, \{\mathcal{X}_3 > T\}$.

(**b**) Show that $P\{\mathcal{Y} > T\} = 3\exp(-2\lambda T) - 2\exp(-3\lambda T)$, and use this result to find $\mathsf{E}[\mathcal{Y}]$, the *average lifetime* of the TMR system. [Hint: $\mathsf{E}[\mathcal{Y}] = \int_0^\infty P\{\mathcal{Y} > T\}\,dT$.] The average lifetime is also known as the mean time before failure (MTBF) or mean time to failure (MTTF) in the reliability literature.

(**c**) Find the *median* value of $\mathcal{Y}$ by solving the equation $P\{\mathcal{Y} > T\} = \frac{1}{2}$ for T.

(**d**) Compare your answers of parts (b) and (c) to the MTBF λ^{-1} and the median lifetime $\lambda^{-1}\ln 2$ for a single module. Do the answers surprise you? Is the TMR system a more reliable system as claimed?

(**e**) Now suppose that $\lambda = -\ln 0.999$. What are the numerical values of $P\{\mathcal{X}_1 > 1\}$ and $P\{\mathcal{Y} > 1\}$?

(**f**) I hope you found in part (e) that $P\{\mathcal{X}_1 > 1\} = 0.999$ and so a single module works with 99.9% reliability for at least one unit of time. What is the largest value of T for which $P\{\mathcal{Y} > T\} \geq 0.999$? How does the TMR system compare to a single module in terms of providing 99.9% reliability over long periods of time?

2. Let $\mathcal{X}$ denote a unit Gaussian random variable. Its CDF is $\Phi(u)$.

(**a**) What is the derivative of $\exp(-u^2/2)$? Use this result to compute $\mathsf{E}[|\mathcal{X}|]$.

(**b**) $Q(x) = \displaystyle\int_x^\infty (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{u^2}{2}\right) du = P\{\mathcal{X} > x\}$ is called the *complementary* CDF. A useful bound is $Q(x) \leq \frac{1}{2}\exp(-x^2/2)$ for $x \geq 0$. Derive this bound by first proving that $t^2 - x^2 > (t-x)^2$ for $t > x > 0$ and then applying this to

$$\exp(x^2/2)Q(x) = \int_x^\infty (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{t^2 - x^2}{2}\right) dt.$$

3. Let $\mathcal{X}$ denote a Gaussian random variable with mean -10 and variance $\sigma^2 = 4$. You have at your disposal two calculators: one can calculate $\Phi(x)$ for $x \geq 0$, and the other can calculate $Q(x)$ for $x \geq 0$. Both have the usual assortment of basic arithmetic functions. Write down expressions for calculating each of the following probabilities with each of the calculators. Remember that the argument of Φ or Q must be ≥ 0 in all cases.
(**a**) $P\{\mathcal{X} < 0\}$. (**b**) $P\{-10 < \mathcal{X} < 5\}$. (**c**) $P\{|\mathcal{X}| \geq 5\}$. (**d**) $P\{\mathcal{X}^2 - 3\mathcal{X} + 2 > 0\}$.

4. The width of a metal trace on a circuit board is modelled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

(**a**) Traces that fail to meet the requirement that the width be in the range 0.9 ± 0.005 microns are said to be defective. What percentage of traces are defective?

(**b**) A new manufacturing process that produces smaller variations in trace widths is to be designed so as to have no more than 1 defective trace in 100. What is the maximum value of σ for the new process if the new process achieves the goal?

5. The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathcal{V}$ with pdf

$$f_{\mathcal{V}}(u) = 0.5 \exp(-|u|), -\infty < u < \infty.$$

Then, the current $\mathcal{I}$ is also a continuous random variable.

(**a**) What values can $\mathcal{I}$ take on?

(**b**) Find the CDF of $\mathcal{I}$.

(**c**) Find the pdf of $\mathcal{I}$.

6. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.

(**a**) Find the pdf of $\mathcal{Y} = \mathcal{X}^2$.

(**b**) Find the pdf of $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0 \\ -u^2, & u < 0 \end{cases}$.

7. [Give me an A! Give me a D! Give me a converter! What have we got? An A/D converter! Go Team!] A signal $\mathcal{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathcal{Y}$ (where $\mathcal{Y} = \alpha$ if $\mathcal{X} > 0$ and $\mathcal{Y} = -\alpha$ if $\mathcal{X} \leq 0$) is used. Note that $\mathcal{Y}$ is a *discrete* random variable.

(**a**) What is the pmf of $\mathcal{Y}$?

(**b**) The *squared error* in representing $\mathcal{X}$ by $\mathcal{Y}$ is $\mathcal{Z} = \begin{cases} (\mathcal{X} - \alpha)^2, & \text{if } \mathcal{X} > 0, \\ (\mathcal{X} + \alpha)^2, & \text{if } \mathcal{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathcal{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathcal{Z}]$. Use LOTUS to ez-ily calculate $\mathsf{E}[\mathcal{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathcal{Z}]$.

(**c**) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathcal{X}$ to the nearest integer $\mathcal{W}$ in the range 3 to +3. Thus, $\mathcal{W} = 3$ if $\mathcal{X} \geq 2.5$, $\mathcal{W} = 2$ if $1.5 \leq \mathcal{X} < 2.5$, $\mathcal{W} = 1$ if $0.5 \leq \mathcal{X} < 1.5$, $\cdots$, $\mathcal{W} = -3$ if $\mathcal{X} < -2.5$. Note that $\mathcal{W}$ is also a discrete random variable. Find the pmf of $\mathcal{W}$.

(**d**) The output of the A/D converter is a 3-bit 2's complement representation of $\mathcal{W}$. Suppose that the output is $(\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0)$. What is the pmf of $\mathcal{Z}_2$? the pmf of $\mathcal{Z}_1$? the pmf of $\mathcal{Z}_0$? Note that $(1, 0, 0)$ which represents -4 is not one of the possible outputs from this A/D converter.