ECE 413: Problem Set 9

Due:	Wednesday March 30 at the beginning of class.
Reading:	Ross, Chapter 5
Noncredit exercises:	Ross, Chapter 5, Problems 10-19; 21, 22, 24, 31-41
Reminders:	No class on Friday March 18 (time off for first evening hour exam) No class on Monday March 28 (time off for second evening hour exam) Enjoy Spring Break!

This Problem Set contains six problems

1. ["Extra! Extra! Read all about it!"] A newsboy purchases H newspapers for c_2 cents each and sells them for c_3 cents each. He can return unsold papers to the publisher for c_1 cents each. Note that $c_1 < c_2 < c_3$. The daily demand $\mathcal{X}$ for papers is a (nonnegative) integer-valued random variable with pmf $p_{\mathcal{X}}(u)$ and CDF $F_{\mathcal{X}}(u)$.

(**a**) What is the probability that he sells all H newspapers? Express your answer in terms of $p_{\mathcal{X}}(u)$ and also in terms of $F_{\mathcal{X}}(u)$.

(**b**) Let $\mathcal{Z}$ denote the daily profit (in cents) that the newsboy makes. Write an expression for $\mathcal{Z}$ in terms of $\mathcal{X}$ and H.

(**c**) Use LOTUS to write an expression for his *average* daily profit. Your answer will depend on H, so call the expression for the average daily profit the function $g(H)$.

(**d**) The newsboy has been buying H papers for some months and making an average profit $g(H)$ each day. One day, he decides to buy one extra paper. What is the probability that he can sell this extra paper? Show that he makes an average *additional* profit of $A(H) = (c_3 - c_2) - (c_3 - c_1)F_{\mathcal{X}}(H)$.

(**e**) Show that the average additional profit $A(H)$ has the property that

$$\cdots \geq A(H-1) \geq A(H) \geq A(H+1) \geq \cdots$$

that is, on average, each extra newspaper brings in smaller extra profit than the previous one. This is called the law of diminishing returns. [Hint: $F_{\mathcal{X}}(H)$ is a non-decreasing function of H].

(**f**) Show that $A(H) < 0$ for sufficiently large values of H

(**g**) How many papers should he purchase to maximize his average profit?

2. The continuous random variable $\mathcal{X}$ has probability density function

$$f_{\mathcal{X}}(u) = \begin{cases} \alpha(1-u), & 0 \leq u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(**a**) What is the value of α? What is the value of $P\{6\mathcal{X}^2 > 5\mathcal{X} - 1\}$?

(**b**) Find the cumulative distribution function $F_{\mathcal{X}}(u)$. Be sure to specify the value of $F_{\mathcal{X}}(u)$ for all $u, -\infty < u < \infty$.

3. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable X with probability density function

$$f_{\mathcal{X}}(u) = \begin{cases} 5(1-u)^4, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

(**a**) If $C = 0.5$, (i.e., the tank holds 500 gallons) and $\mathcal{X}$ happens to have value 0.68 one particular week, (e.g. 680 people show up each wanting to purchase a gallon of gas for their snowblowers or lawnmowers), can the gas station satisfy the demand that week? That is, can the gas station supply gasoline to all those who want to buy it that week? How about if $\mathcal{X}$ happens to have value 0.43 some other week?

(**b**) If $C = 0.5$, what is the probability that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666..., then, in the long run, the gas station can supply the weekly demand two weeks out of three.

(**c**) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

(**d**) Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let $\mathcal{Y}$ denote the amount of gasoline sold per week. How is $\mathcal{Y}$ related to $\mathcal{X}$, the weekly demand for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!) What is the *average* weekly gross profit?

(**e**) Suppose that the owner pays \$$20C$ as weekly rent on a tank of capacity $1000C$ gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the average weekly net profit and what value of C maximizes the average weekly net profit?

4. $\mathcal{X}$ is a continuous random variable with pdf $f_{\mathcal{X}}(u) = 0.5\exp(-|u|), -\infty < u < \infty$.

(**a**) What is the value of $P\{\mathcal{X} \le \ln 2\}$?

(**b**) Find the *conditional* probability that $\{|\mathcal{X}| \le \ln 2\}$ *given* that $\{\mathcal{X} \le \ln 2\}$.

(**c**) Find the numerical value of $P\{\cos(\pi\mathcal{X}/2) < 0\}$.

5. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.

(**a**) If $\mathcal{Y} = \mathcal{X}^2$, what are the mean and variance of $\mathcal{Y}$?

(**b**) If $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \ge 0 \\ -u^2, & u < 0 \end{cases}$ use LOTUS to find $\mathsf{E}[Z]$

(**c**) On a completely unrelated LOTUSian question, if $\mathcal{U}$ is a geometric random variable with parameter $\frac{1}{2}$, and $\mathcal{V} = \sin(\pi\mathcal{X}/2)$, what is the value of $\mathsf{E}[\mathcal{V}]$?

6. Consider a sphere whose radius is a random variable $\mathcal{R}$ with pdf $f_{\mathcal{R}}(u) = 2u,\ 0 < u < 1$, and 0 otherwise.

(**a**) What is the average radius of the sphere? What is the average volume? What is the average surface area? If a sphere of average radius is called an *average sphere*, then does an average sphere have the average volume? Does it have the average surface area?

(**b**) Show that $\mathsf{E}[\mathcal{R}] > \mathsf{E}[\mathcal{R}^2] > \mathsf{E}[\mathcal{R}^3]$ for *any* pdf for $\mathcal{R}$ that is nonzero only on the unit interval $(0, 1)$.