University of Illinois Spring 2005

ECE 413: Problem Set 6

Due: Wednesday March 9 at the beginning of class.
Reading: Ross, Chapters 3, 4, 5, and the notes on decision-making on the class web page

This Problem Set contains five problems

1. In Problem 3 of Hour Exam I, we considered a tennis game in which A and B win points with probabilities p and $q = 1 - p$. If the score reaches deuce (3-3) (an event D of probability $20p^3q^3$), then A and B won the game with (conditional) probabilities $p^2/(p^2 + q^2)$ and $q^2/(p^2 + q^2)$ respectively. The game can end without the score ever reaching deuce with scores of 4-0, 4-1, and 4-2 winning for A and 0-4, 1-4, and 2-4 winning for B.

(**a**) What are the probabilities that the scores reach these 6 possible terminal values? (Hint for those whose drug of choice is MTV and not ESPN: the player who wins the last point in a game also wins the game. Thus, the probability of A winning 4-2 is *not* $\binom{6}{2}p^4q^2$.)

(**b**) What is the probability that A wins the game? Denote this result as $f(p)$.

(**c**) A little thought will show, I hope!, that B wins the game with probability $f(1-p)$. Explain from first principles why $f(p)$ should have the following properties and determine if *your* answer to part (b) satisfies these properties:
$f(0) = 0, f(0.5) = 0.5, f(1) = 1$. Does this mean that $f(p)$ is a linear function of p?

(**d**) Find the first two terms of the Taylor series for $f(p)$ in the neighborhood of $p = 0.5$ and determine the effect on the probability of A winning the game if $p = 0.5 + \epsilon$ where ϵ is a small number.

2. The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides independently of the other groups whether to support or oppose the motion. All members of the group then vote in accordance with the caucus decision. If you believe that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a real bargain price . . .

(**a**) Let A, B, C, and D respectively denote the events that the four groups vote to eliminate all income taxes on capital gains. Suppose that the probabilities of these independent events are $P(A) = 0.9, P(B) = 0.6, P(C) = 0.5$ and $P(D) = 0.2$. What is the probability that the bill passes?

(**b**) The President vetoes the bill as a budget-breaker. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities $P(E) = 0.99, P(F) = 0.4, P(G) = 0.6$, and $P(H) = 0.1$, what is the probability that the motion to override the veto passes ?

Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

3. Consider the matrix of Problem 4 of Problem Set 5 as a likelhood matrix. The three hypotheses are that the transmitted signal $\mathcal{X}$ took on values 1, 2, or 3 and the receiver observes that $\mathcal{Y}$ took on values 1, 2, or 3.

(**a**) Having observed $\mathcal{Y}$, what is the receiver's maximum-likelihood decision rule as to which signal was transmitted ?

(**b**) The receiver knows the pmf of $\mathcal{X}$. What is the receiver's maximum *a posteriori* probability (MAP) decision rule?

4. We return to the baseball pitcher of Problem 3 of Problem Set 6. A fan sitting in the bleachers observes that the batter got a hit (the event H), but is too far away to be able to tell what kind of pitch it was.

(**a**) What is his *maximum-likelihood* decision rule as to whether the pitch was a fast ball, curve ball or slider?

(**b**) Now suppose that after cheering the hit, the fan returns to his seat and finds $P(H) = 0.25$ listed in the program guide. Having lasted in ECE 413 through Problem Set 6, he can compute $P(F), P(C)$ and $P(S)$. But, since he dropped the course immediately after Hour Exam I, please help him compute his maximum *a posteriori* probability decision rule as to what kind of pitch it was.

5. ["Give me an F!" shouted the cheerleader...] H_0, H_1, and H_2 respectively denote the hypotheses that a student is excellent, good, or average (there are no poor students). The number of grade points earned by the student in a course is a random variable $\mathcal{X}$ that takes on values 3, 6, 9, and 12 only. The professor knows that the pmf of $\mathcal{X}$ when H_0 is true is $p_0(12) = 0.75, p_0(9) = 0.15, p_0(6) = 0.08, p_0(3) = 0.02$, that is, an excellent student has 75% chance of doing well enough on the exam to get an A, 15% chance of a B, etc. Similarly, when H_1 is the true hypothesis, the pmf of $\mathcal{X}$ is $p_1(12) = 0.15, p_1(9) = 0.6, p_1(6) = 0.15, p_1(3) = 0.1$, while if H_2 is true, $p_2(12) = 0.05, p_2(9) = 0.1, p_2(6) = 0.65, p_2(3) = 0.2$. The professor observes $\mathcal{X}$ and must decide which of the hypotheses H_0, H_1, H_2 is true.

(**a**) What is the professor's maximum-likelihood decision rule?

(**b**) What is the probability that an excellent student is mistakenly labeled as good? What is the probability that an excellent student is mistakenly labeled as average? What is th probability that an average student is classified either as good or as excellent?

(**c**) If $P(H_0) = 0.2, P(H_1) = 0.55$, and $P(H_2) = 0.25$, what is the probability that the maximum-likelihood decision rule mis-classifies students?

(**d**) What is the Bayes decision rule corresponding to these probabilities and what is the probability that the Bayes decision rule mis-classifies students?

(**e**) At the Lake Wobegon campus of the University, 95% of students are excellent and 5% are good (and thus they are all above average!) What is Bayes decision rule in this case? That is, what does the Bayesian professor decide about a student based on the four possible results of the students exam?