University of Illinois Spring 2005

ECE 413: Problem Set 4

Due:	Wednesday February 16 at the beginning of class.
Reading:	Ross, Chapter 4
Noncredit Exercises:	DO NOT turn these in.
	Chapter 4: Problems 34, 35, 38, 39, 40-43, 48, 51-59;
	Theoretical Exercises 16-18; Self-Test Problems 9, 13, 15, 16.

This Problem Set contains six problems

1. Eight persons hold reservations for travel in a 5-passenger limousine from Champaign to St. Louis. The number of persons who actually show up to travel can be modeled as a binomial random variable $\mathcal{X}$ with parameters $(8, 0.5)$. Naturally, if more than 5 persons show up, only 5 get to go and the rest are left behind. Let $\mathcal{Y}$ denote the number of persons left behind.

(**a**) What is $\mathsf{E}[\mathcal{X}]$?

(**b**) Find the pmf of $\mathcal{Y}$.

(**c**) What is $\mathsf{E}[\mathcal{Y}]$? Calculate this in two ways: (i) from your answer to part (b), and (ii) by using the fact that $\mathcal{Y}$ is a function of $\mathcal{X}$, and hence LOTUS allows us to calculate $\mathsf{E}[\mathcal{Y}]$ directly from the pmf of $\mathcal{X}$.

2. In the game of Chuck-A-Luck played at fairs and carnivals in the MidWest, bets are placed on numbers 1, 2, 3, 4, 5, 6, and then three fair dice are rolled. If the number chosen does not show up on any of the three dice, the bettor loses his stake. Otherwise, the dealer pays the bettor one or two or three times the amount staked according as the number chosen shows up on one or two or all three of the dice. Of course, the amount of the bet is also returned to the bettor but is *not* counted as part of the *winnings* from this game. Let $\mathcal{X}$ denote the winnings in this game for a \$6 bet, and remember that negative values of $\mathcal{X}$ correspond to losses.

(**a**) What are the values taken on by $\mathcal{X}$?

(**b**) What is the pmf of $\mathcal{X}$?

(**c**) What is the value of $\mathsf{E}[X]$?

(**d**) A player splits his \$6 bet and wagers \$1 on each of the six numbers. Let $\mathcal{Y}$ denote the winnings of this player. Repeat parts (a)-(c) for $\mathcal{Y}$. Does the splitting strategy improve the average winnings in this game?

3. Let $\mathcal{Y}$ denote a Poisson random variable with parameter λ.

(**a**) Show that $P\{\mathcal{Y} \text{ is even}\} = \exp(-\lambda)\cosh(\lambda)$.

(**b**) In Problem 6 of Problem Set 3, you proved (I hope!) that the probability that a binomial random variable $\mathcal{X}$ with parameters (N, p) is *even* is $[1 + (1 - 2p)^N]/2$. Now, for large N and small p, the binomial probability $P\{\mathcal{X} = k\}$ is well approximated by the Poisson probability $\exp(-Np)(Np)^k/k!$. Under the same conditions, show that $[1 + (1 - 2p)^N]/2 \approx \exp(-Np)\cosh(Np)$ and thus your answer of part (a) is consistent with the previous result.

(c) Now suppose that the value of λ is unknown. The experiment is performed and it is observed that $\mathcal{Y} = k$. What is the *likelihood* of this observation? What is the *maximum likelihood* estimate $\hat{\lambda}$ of λ? That is, what choice of positive number $\hat{\lambda}$ maximizes the likelihood of the observation $\mathcal{Y} = k$?

4. Suppose that 105 passengers hold reservations for a 100-passenger flight from Chicago to Champaign. The number of passengers who show up at the gate can be modeled as a binomial random variable $\mathcal{X}$ with parameters $(105, 0.9)$.

(a) On average, how many passengers show up at the gate?

(b) If $\mathcal{X} \leq 100$, everyone who shows up gets to go. Find the value of $P\{\mathcal{X} \leq 100\}$.

(c) Explain why the number of *no-shows* can be modeled as a binomial random variable $\mathcal{Y}$ with parameters $(105, 0.1)$.

(d) Notice that the probability that everyone who shows up gets to go can also be expressed as $P\{\mathcal{Y} \geq 5\}$. Use the *Poisson approximation* to compute $P\{\mathcal{Y} \geq 5\}$ and compare your answer to the "more exact" answer that you found in part (b).

5. There are N multiple-choice questions (with 5 possible answers each) on a certain exam. A student knows the answers to K questions and answers them correctly. On the remaining $N - K$ questions, the student guesses randomly among the 5 choices. The examiner knows N, and can observe the values of $\mathcal{C}$, the number of *c*orrect answers, and $\mathcal{W} = N - \mathcal{C}$, the number of *w*rong answers on the answer sheet. Note that $\mathcal{C}$ can have values K, $K+1$, $\ldots$, N. What the examiner is really interested in, though, is *estimating* the value of K.

(a) Explain why it is reasonable to model $\mathcal{W}$ as a binomial random variable with parameters $(N - K, 0.8)$. What assumptions are you making?

(b) Suppose that n answers are incorrect, that is, $\mathcal{W} = n$ and $\mathcal{C} = N - n$. What is the *likelihood* of this observation? Hint: your answer will depend on N, n and the unknown parameter K that the examiner is interested in estimating.

(c) Having observed that $\mathcal{W} = n$, the examiner is sure that K cannot exceed $N - n$, i.e., K can have value $0, 1, 2, \ldots, N-n$ only. Use the method of Proposition 6.1 of Chapter 4 in Ross to show that the likelihood you found in part (b) is maximized at $\hat{K} = \lfloor N - 1.25n + 1 \rfloor$.

(d) Since $\mathcal{C} = N - n$, a *guessing penalty* is applied by subtracting $\lfloor 0.25n \rfloor$ from $\mathcal{C}$ to get an estimate of K. For $N = 100$ and $K = 90$, compare the *examiner's estimate* $\tilde{K} = N - n - \lfloor 0.25n \rfloor$ and the maximum likelihood estimate $\hat{K}$ for $n = 0, 1, \ldots 10$. Notice that lucky guesses cause the examiner to overestimate K while the unlucky student who blows all ten problems has to suffer the further indignity of having the score reduced to something smaller than K.

(e) [Noncredit exercise] If think that the result of part (e) is grossly unfair, write a letter to the Educational Testing Service complaining about the guessing penalty.

6. Ross, Problem 4 on page 171.