University of Illinois Spring 2005

ECE 413: Problem Set 2

Due: Wednesday February 2 at the beginning of class.
Reading: Ross, Chapter 1 and Chapter 2, Sections 1-5
Noncredit Exercises: DO NOT turn these in.

Chapter 1: Problems 1-5, 7, 9; Theoretical Exercises 4, 6, 13; Self-Test Problems 1-15.
Chapter 2: Problems 3, 4, 9, 10,11-14; Theoretical Exercises 1-3, 6, 7, 10, 11, 2,16, 19, 20; Self-Test Problems 1-8

This Problem Set contains six problems

1. The binomial theorem avers that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is called a *binomial coefficient.*

(a) Show that $\binom{n}{k} = \binom{n}{n-k}$ and that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(b) $\binom{n}{k}$ is also the number of events of size k defined on a sample space of size n. Use this notion to prove the results of part (a) *without using the binomial theorem or any explicit formula for* $\binom{n}{k}$.

(c) Suppose that n is fixed. What choice(s) of k *maximize* the value of $\binom{n}{k}$? For which values of n is the maximum unique? If there is more than one maximum, how many maxima can there be? *Warning:* You cannot use the usual calculus trick of differentiating here because k is not a continuous variable.

(d) Write down the first five terms in the expansions of $(1+x)^n$ and $(1-x)^n$ (preferably on two consecutive lines and with coefficients of x^k on the two lines being one above the other in vertical alignment.) Now show that exactly 2^{n-1} of the 2^n subsets of a sample space of size n contain an even number of elements (the other 2^{n-1} subsets contain an odd number of elements). Hint: remember that 0 is an even number and set $x = 1$ in $(1+x)^n \pm (1-x)^n$.

2. **(a)** What is the derivative of $\exp(-x^2/2)$?

(b) Find the value of $\int_0^\infty x \exp(-x^2/2)\, dx$. Hint: read your answer of part (a).

(c) Find the value of $\int_0^\infty \int_0^\infty \exp(-[x^2+y^2]/2)\, dx\, dy$. Hint: convert to polar coordinates and read your answer to part (b).

3. **(a)** An ice cream manufacturer makes unflavored ice cream and then creates "specialty flavors" by blending in one or more of the five essences: vanilla, chocolate, fudge, mint, and almond into the ice cream. How many specialty flavors can the manufacturer create? **Optional noncredit exercise:** identify the manufacturer!

(b) Consider events A and B defined on a sample space. If the probability that at least one of the two events occurred is 0.6 and the probability that at least one of the events did *not* occur is 0.8, what is the probability that *exactly one* of the two events occurred?

4. An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

(a) Let A denote the event that the least significant bit is a ONE. What is $P(A)$?

(b) Let B denote the event that the register contains 5 ONEs and 3 ZEROes. What is $P(B)$?

(c) What is $P(A \cup B)$? What is $P(A \cap B)$? What is the probability that exactly one of A and B occur, i.e. what is $P(A \oplus B)$?

(d) Let C denote the event that there are 4 ONEs and 4 ZEROes in the register. What is $P(C)$?

5. Some years ago, the government of a small island in the North Atlantic decided to change from a constitutional hereditary monarchy to a constitutional appointed monarchy. Six applicants Andy, Beth, Chuck, Di, Eddie, and Fergie presented themselves before the House of Commons Search Committee for interviews. Subsequently, the Committee forwarded a *short list* of three names to the whole House of Commons for its decision.

(a) How many different short lists could the Committee have selected?

(b) Assume that all short lists were equally likely to have been chosen.

i. What is the probability that Beth is on the short list?

ii. What is the probability that Chuck is on the short list?

iii. What is the probability that both Beth and Chuck are on the short list?

iv. What is the probability that Chuck and two women are on the short list? (Those who wish to pretend that they don't read the National Enquirer should note that Beth, Di, and Fergie are women).

6. The manufacturer of a cereal tests samples from the production line to see if the samples snap, crackle, and pop as advertised. Let A, B, and C respectively denote the events that the sample being tested *does not* snap, *does not* crackle, and *does not* pop. The manufacturer's tests show that $P(A) = P(C) = 0.3$, $P(B) = 0.2$, $P(AB \cup AC \cup BC) = 0.3$, $P(A \cap B \cap C) = 0.05$, $P(A \cap B) = 0.1$, and $P(A \cap C) = 2P(B \cap C)$.

(a) Sketch the sample space and indicate on it the events A, B, and C.

(b) What is the probability that the cereal snaps, crackles, and pops?

(c) Cereal that fails exactly one test is sold to discount supermarket chains to be marketed under the names Soggies, Blecchies, and Mushies. What is the probability that the sample fails *only* the snap test? *only* the crackle test? *only* the pop test?