ECE 413: Solutions to Problem Set 6

1. (**a**) Beth is on $\binom{5}{2} = 10$ of the 20 possible short lists, and hence $P(\text{Beth chosen}) = \dfrac{1}{2}$.

(**b**) Chuck is also on 10 lists, but 4 of those 10 lists include Beth. Three of the other 6 lists include Di, and his conditional probability of being chosen is $\frac{1}{2}$. On the other 3 lists, his conditional probability of being chosen is $\frac{1}{3}$. Hence,

$$P(\text{Chuck chosen}) = 0 \times \frac{4}{20} + \frac{1}{2} \times \frac{3}{20} + \frac{1}{3} \times \frac{3}{20} = \frac{1}{8}.$$

An alternative solution to this is instructive. Beth and Di are chosen with probabilities $\frac{1}{2}$ and 0 respectively. The other four must necessarily share the remaining probability equally because any argument that purports to show that Chuck has a better chance than Eddie (say) can be modified (by interchanging the two names everywhere) into an argument that shows that Eddie has a better chance than Chuck. Therefore, $P(\text{Chuck chosen}) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

(**c**) Given that Beth was chosen as the monarch, the short list must have been one of the 10 with Beth on it. Four of these also had Di on it, giving $P(\text{Di on list} \mid \text{Beth chosen}) = \frac{4}{10} = \frac{2}{5}$. In fact, all the other five have equal chance $\frac{2}{5}$ of being on the short list with Beth. (Exercise: Explain why the probabilities add up to 2)

2. If the i-th and the n-th trials both resulted in successes, then the remaining $r-2$ successes must have been scattered among the other $n-2$ trials. Thus,

$$\begin{aligned} P\{i\text{-th trial is a success} \mid \mathcal{X} = n\} &= \frac{P\{i\text{-th trial is a success} \cap \{\mathcal{X} = n\}\}}{P\{\mathcal{X} = n\}} \\ &= \frac{p \cdot p \binom{n-2}{r-2} p^{r-2}(1-p)^{n-r}}{p \cdot \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r}} = \frac{r-1}{n-1}. \end{aligned}$$

Alternatively, *conditioned on* $\mathcal{X} = n$, the number of successes on the first $n-1$ trials is a *binomial* random variable $\mathcal{Y}$ with parameters $(n-1, p)$, and we know that $\mathcal{Y} = r-1$ has occurred. Conditioned on this, the probability that the i-trial resulted in a success is just $(r-1)/(n-1)$ as we showed in class.

3. Since the pitcher can only pitch fast balls, curve balls, and sliders, we have that $P(F)+P(C)+P(S) = 1$, and since $P(C) = 2P(F)$, we conclude that $P(S) = 1 - 3P(F)$. We are also told that

$$\begin{aligned} P(H) = \frac{1}{4} &= P(H|F)P(F) + P(H|C)P(C) + P(H|S)P(S) \\ &= P(H|F)P(F) + P(H|C) \cdot 2P(F) + P(H|S)(1 - 3P(F)) \\ &= \frac{2}{5}P(F) + \frac{1}{4} \cdot 2P(F) + \frac{1}{6}(1 - 3P(F)) \end{aligned}$$

from which we get that $P(F) = \dfrac{5}{24}, P(C) = \dfrac{10}{24}$, and $P(S) = \dfrac{9}{24}$.

4. Let A denote the event that your initial choice is the curtain concealing the car and B the event that your final choice is the curtain concealing the car. Clearly $P(A) = \frac{1}{3}$. Now the value of $P(B|A)$ and $P(B|A^c)$ (and therefore $P(B)$) depends on your *strategy* in response to Monty's blandishments.

(**a**) Suppose that you always switch. Then, obviously $P(B|A) = 0$ since you chose curtain with the car initially and are now choosing the other curtain (with the goat.) Also, $P(B|A^c) = 1$ since you had a goat initially, and you know where the other goat is. Thus, $P(B) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$.

(**b**) If you always stay put, then $P(B|A) = 1$ and $P(B|A^c) = 0$ leading to $P(B) = \frac{1}{3}$.

(**c**) If, say, you toss a fair coin (independently!) to decide whether to stay put (H) or switch (T), then $P(B|A,H) = 1, P(B|A,T) = 0$ and therefore $P(B|A) = P(B|A,H)P(H) + P(B|A,T)P(T) = \frac{1}{2}$. Similarly $P(B|A^c) = \frac{1}{2}$. Monty is correct in his assertion. Besides, he said it on national TV! He wouldn't lie to you on national TV, would he?

Many students are puzzled by the results of this problem. Consider that you have a $\frac{1}{3}$ chance of choosing the curtain with the car in the first place. If you never switch, you win the car. The probability that one of the *other* two curtains is concealing the car is $\frac{2}{3}$. Monty in essence is asking "Would you rather have what's behind both those two other two curtains except you don't *have* to take any goats home with you unless you really want to, and oh, by the way, here is one goat which is not telling you anything new since you knew that there was at least one goat there."

Exercise: What would be your chances of winning if you tossed a biased coin?

(**d**) It makes no difference whether you or your friend chooses first; you have equal probability of choosing the curtain with the car. He just happens to have been unlucky. In this game, but you should stay put because the chances are $\frac{2}{3}$ that the car is behind one of the two curtains picked by you and your friend. He's already gotten the goat, and so you get the car with probability $\frac{2}{3}$.

5. This game is different from the one in Problem 4 in that you have no idea what the rules of the game are. If the man is intent on separating you from your money as quickly as possible, he will not offer the chance to switch unless you picked the shell hiding the pea in the first place! That is, if you picked the wrong shell, the man will reveal the pea and you will lose your bet. Of course, if you look like a person willing to play several rounds, the man may set you up by playing by Monty's rules (and allowing you to win with probability 2/3) for some time. Then you will place a large bet on the wrong shell, and all of a sudden, you will not be given the choice of changing your bet! Personally, I would stick with the shell originally chosen since it gives me at least a 1/3 probability of winning regardless of the man's strategy; your experience (and monetary losses) may vary

6. (**a**) For the single gigantic car, each part fails with probability p. All the N parts of a given type must fail in order for the car to fail, and this occurs with probability p^N, and so the probability that at least one part of a given type is working is $1 - p^N$. The probability that at least one part of each of the M types is working is $(1 - p^N)^M$, and hence the probability of system failure is $1 - (1 - p^N)^M$.

On the other hand, if we have N cars each with M parts, then the probability that at least one part fails in a car is $1 - (1 - p)^M$. This is also the probability that the car itself has failed. But, we have N cars, and thus the probability that at least one car is in working condition (i.e. that not all have failed) is $(1 - (1 - p)^M)^N$.

(**b**) For the single gigantic car, we want $1 - ((1 - 0.2^N)^5 < 10^{-3}$ which is achieved by choosing $N \geq 6$. For the N small cars, we want $(1 - (1 - 0.2)^5)^N < 10^{-3}$ which is achieved by choosing $N \geq 18$. Thus, a gigantic car with 6 engines, 6 transmissions, 6 brakes etc provides more reliable transmission than having 17 different compact cars.

(**c**) If $M = 1000$, we get that $N \geq 9$ for the gigantic car while typical calculators blow up on the calculation for the number of small cars needed. The Unix high-precision utility *bc* gives that $1 - (1 - 0.2)^{1000} = 0.99\ldots 9986160344\ldots$ where there are eightyseven 9's preceding the 8616, and the number of cars needed is astronomical. Parking, of course, is impossible, but keep in mind that a gigantic car with 9 engines etc will also be very difficult to park!

Moral: It is better to replicate parts than to replicate systems.