1. (a) $A$ and $B$ are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.
   
   TRUE $P(A \cup B) \geq \max\{P(A), P(B)\}$
   
   FALSE $P(A \cap B) \geq \min\{P(A), P(B)\}$ (as $P(A \cap B)$ is always less than or equal to $P(A)$ and $P(B)$)
   
   FALSE $P(A|B) + P(A|B^c) = 1$.
   
   TRUE $P(A|B)P(B) + P(A^c|B)P(B^c) = 1 - P(A \oplus B)$.
   
   FALSE $P(A|B)P(B) + P(A^c|B^c)P(B^c) = P(A)$, (true if RHS were $P(B)$)
   
   TRUE If $P(A) = P(B)$, then $P(A|B) = P(B|A)$.
   
   FALSE If $P(A|B) = P(B|A)$, then $P(A) = P(B)$. Conclusion does not hold if $P(A|B) = P(B|A) = 0$.
   
   TRUE If $P(A|B) = P(A)$, then $P(B^c|A) = 1 - P(B)$. $A$ and $B$ are independent events.

(b) $X$ and $Y$ are random variables such that $\text{var}(X) = \text{var}(Y) = \sigma^2 < \infty$.

$\text{var}(2X + 3Y + 4) = 4\sigma^2 + 9\sigma^2 + 12 \cdot \text{cov}(X, Y) = \text{var}(3X - 2Y + 1) = 9\sigma^2 + 4\sigma^2 + 12 \cdot \text{cov}(X, Y)$ implies that $\text{cov}(X, Y) = 0$.

TRUE $X$ and $Y$ are uncorrelated random variables.

FALSE $X$ and $Y$ are independent random variables.

TRUE $\text{var}(2X + 3Y + 4) = \text{var}(2X - 3Y + 1)$.

TRUE $\text{cov}(X + Y, X - Y) = 0$.

2. (a) Let $A_n$ denote the event that the mailman is not bitten on the $n$-th day. Then,
   
   $\{X = n\} = A_n \cap A_{n-1} \cap A_{n-2} \ldots \cap A_1$ and $\{X > n\} = A_n \cap \neg A_{n-1} \cap A_{n-2} \ldots \cap A_1$. We are given that
   
   $P\{A_n^c|A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \frac{1}{n+1}$ and hence $P\{A_n|A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \frac{n}{n+1}$.

Note also that $P\{A_1\} = P\{A_1^c\} = \frac{1}{2}$. Therefore,

$P\{X > n\} = P\{A_n \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \prod_{i=1}^n \frac{1}{i+1} = \frac{n}{n+1}$.

Hence $p_X(n) = P\{X = n\} = P\{X > n - 1\} - P\{X > n\} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, $n = 1, 2, \ldots$.

(b) $E[X] = \sum_{n=1}^{\infty} n \cdot P\{X = n\} = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$ from the fact that the harmonic series diverges. Alternatively, since $X$ takes on positive integer values, we have from Problem 4(a) of Problem Set 8 that $E[X] = \sum_{n=0}^{\infty} P\{X > n\} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$.

3. The component lifetimes are exponential random variables with parameter $\lambda$. Thus, the probability that a component fails before time $T$ is $1 - \exp(-\lambda T)$. Since the failures are independent, we have that $P\{X \leq T\} = [1 - \exp(-\lambda T)]^3$ while $P\{X > T\} = 3 \cdot \exp(-\lambda T) - 3 \cdot \exp(-2\lambda T) + \exp(-3\lambda T)$. We can thus write

$E[X] = \int_0^{\infty} P\{X > T\} dT = \int_0^{\infty} 3\exp(-\lambda T) - 3\exp(-2\lambda T) + \exp(-3\lambda T) dT = \frac{3}{\lambda} - \frac{3}{2\lambda} + \frac{1}{3\lambda} = \frac{11}{6\lambda}$.

4. For $-1 < u < 1$, the likelihood ratio is $L(u) = f_1(u)/f_0(u) = 1 - |u|/1^2 = 2 - 2|u|$.

(a) When $X = u$ is the observation, the maximum-likelihood decision rule decides in favor of $H_1$ if $L(u) > 1$. Hence $\Gamma_1 = \{u : |u| < \frac{1}{2}\}$ and $\Gamma_0 = \{u : \frac{1}{2} < |u| < 1\}$.

(b) $P_{\{\cdot\}} = \int_{\Gamma_1} f_0(u) du = \int_{-\frac{1}{2}}^{\frac{1}{2}} du = \frac{1}{2}$, $P_{\{\cdots\}} = \int_{\Gamma_0} f_1(u) du = 2 \int_{\frac{1}{2}}^{1} (1 - u) du = 2 \left[ u - \frac{u^2}{2} \right]_{\frac{1}{2}}^{1} = \frac{1}{4}$. 
When \( X = u \) is the observation, the MAP decision rule decides in favor of \( H_1 \) if \( \Lambda(u) > \pi_0/\pi_1 \). Hence, \( \Gamma_1 = \{ u : |u| < \frac{5}{6} \} \) and \( \Gamma_0 = \{ u : \frac{5}{6} < |u| < 1 \} \) for the MAP decision rule. We get

\[
P_{e_A} = \int_{\Gamma_1} f_0(u) \, du = \int_{-\frac{5}{6}}^{\frac{5}{6}} \frac{1}{2} \, du = \frac{5}{6} \quad \text{and} \quad P_{MD} = \int_{\Gamma_0} f_1(u) \, du = 2 \int_{\frac{5}{6}}^{1} (1-u) \, du = 2 \left( 1 - \frac{u^2}{2} \right) \bigg|_{\frac{5}{6}}^{1} = \frac{1}{36}.
\]

Hence, \( P_e = \pi_0 \cdot P_{e_A} + \pi_1 \cdot P_{MD} = \frac{1}{4} \times \frac{5}{6} + \frac{3}{4} \times \frac{1}{36} = \frac{5}{24} + \frac{1}{48} = \frac{11}{48} \).

The MAP decision rule compares the likelihood ratio \( \Lambda(u) = 2 - 2|u| \) and decides in favor of \( H_0 \) if \( \Lambda(u) < \pi_0/\pi_1 \). Since \( \Lambda(u) \) takes on values in \( (0, 2] \), it will never exceed \( \pi_0/\pi_1 \) if \( \pi_0 > \frac{2}{3} \). Thus, the MAP decision rule always decides in favor of \( H_0 \) if \( \pi_0 > 2/3 \), and achieves average error probability \( P_e = \pi_1 < 1/3 \).

On the other hand, there is no value of \( \pi_0, 0 < \pi_0 < 1 \) for which the MAP rule always decide in favor of \( H_1 \), because no matter how large \( \pi_1 \) is, the ratio \( \pi_0/\pi_1 \) will be smaller than the threshold, and the decision will be in favor of \( H_0 \).

5. (a) The inter-arrival time in a Poisson process with arrival rate \( \lambda \) (time between two successive chalk breaks on this instance) is an exponential random variable with parameter \( \lambda \). Hence, the average length of time between successive chalk-breaks is the mean of this exponential random variable, which is \( \lambda^{-1} = 10 \) minutes.

(b) The number of times that the professor breaks the chalk during a 50 minute lecture is a Poisson random variable \( N(0, 50) \) with parameter \( \lambda 	imes 50 = 5 \) and mean value \( E[N(0, 50)] = 5 \).

(c) From part (b), we get that \( P\{N(0, 50) = 6\} = \frac{5^6}{6!} \exp(-5) \). Now, for \( 0 \leq k \leq 6 \),

\[
P\{\{N(0, 25) = k\} \mid N(0, 50) = 6\} = \frac{P\{\{N(0, 25) = k\} \cap \{N(0, 50) = 6\}\}}{P\{N(0, 50) = 6\}}
\]

\[
= \frac{P\{\{N(0, 25) = k\} \cap \{N(25, 50) = 6 - k\}\}}{P\{N(0, 50) = 6\}}
\]

\[
= \frac{P\{N(0, 25) = k\} \cdot P\{N(25, 50) = 6 - k\}}{P\{N(0, 50) = 6\}}
\]

\[
= \frac{(2.5)^k \exp(-2.5) \times (2.5)^{6-k} \exp(-2.5)}{\frac{5^6}{6!} \exp(-5)}
\]

\[
= \left( \frac{6}{k} \right) \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^{6-k}
\]

Thus, the conditional pmf of \( N(0, 25) \) given that \( \{N(0, 50) = 6\} \) is a binomial pmf with parameters \((6, 1/2)\) and hence the expected value is 3.

6. (a) \( f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) \, dv = \int_0^1 u + v \, dv = uv + \frac{v^2}{2} \bigg|_0^1 = u + \frac{1}{2} \), for \( 0 < u < 1 \), and \( f_X(u) = 0 \) otherwise.

(b) It is obvious from the symmetry of the problem that \( f_Y(v) = v + 0.5 \) for \( 0 < v < 1 \). We readily see that \( f_{X,Y}(u, v) \neq f_X(u) f_Y(v) \) and hence \( X \) and \( Y \) are dependent random variables

\[
f_{Y|X} \left( v \left| \frac{1}{3} \right. \right) = \frac{f_{X,Y} \left( \frac{1}{3}, v \right)}{f_X \left( \frac{1}{3} \right)} = \frac{\frac{1}{3} + v}{\frac{1}{3}} = \frac{2}{3} + \frac{6}{5}v \text{ for } 0 < v < 1, \text{ and } f_{Y|X} \left( v \left| \frac{1}{3} \right. \right) = 0 \text{ otherwise.}
\]

7. The impulse response is a damped oscillation if the roots of \( s^2 + As + B \) are complex numbers, that is, if \( A^2 < 4B \). We have \( P\{A^2 < 4B\} = \int_{u=0}^{1} \int_{v=u^2/4}^{1} 1 \, dv \, du = \int_{0}^{1} 1 - \frac{u^2}{4} \, du = u - \frac{u^3}{12} \bigg|_0^1 = \frac{11}{12} \).

8. \( Z = Y - X \) takes on values in \((0, 1)\). We readily find that for \( 0 < \alpha < 1 \),

\[
1 - F_Z(\alpha) = P\{Z > \alpha\} = P\{Y > X > \alpha\} = P\{Y > X + \alpha\} = 2 \times ((1 - \alpha)^2/2) = (1 - \alpha)^2.
\]

Differentiating, we get \( f_Z(\alpha) = 2(1 - \alpha) \) for \( 0 < \alpha < 1 \), and \( f_Z(\alpha) = 0 \) otherwise.