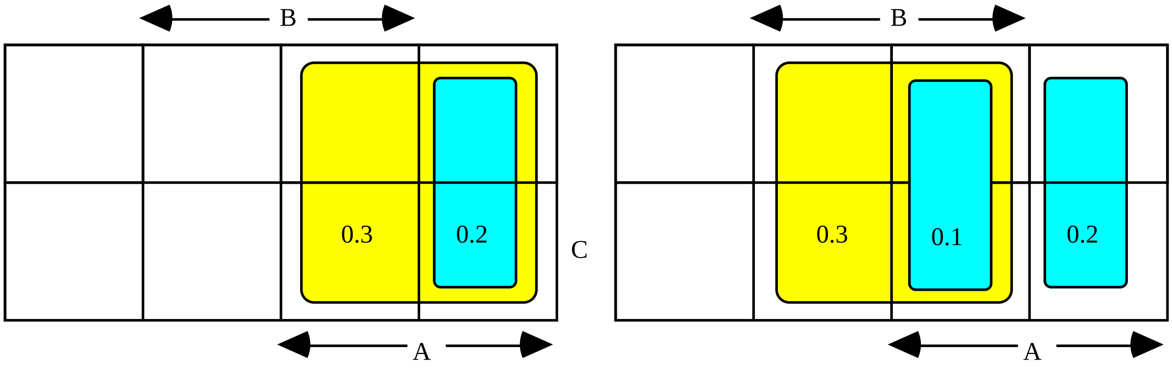
ECE 413: Solutions to Hour Exam I

1. [20 points] Let A, B, and C denote three events defined on a sample space Ω, and suppose that $P(A) = P(B) = 0.3$, $P(C) = 0.5$, and $P(A \cap B^c) = P(A^c \cap B^c \cap C) = 0.2$. Find the following probabilities: $P(A \cap B)$, $P(A^c \cap B)$, $P((A \cup B \cup C)^c)$, and $P(C^c | (A^c \cap B^c))$.

The Karnaugh maps shown below are very useful in visualizing the problem.



We get that

- $P(A \cap B) = P(A) - P(A \cap B^c) = 0.3 - 0.2 = 0.1$. Note that $P(A \cup B) = 0.5$.
- $P(A^c \cap B) = P(B) - P(A \cap B) = 0.3 - 0.1 = 0.2$.
- $P((A \cup B \cup C)^c) = 1 - P(A \cup B \cup C) = 1 - [P(A \cup B) + P(A^c \cap B^c \cap C)] = 1 - [0.5 + 0.2] = 0.3$. Note that $P((A \cup B \cup C)^c) = P(A^c \cap B^c \cap C^c)$, and also that $P(A^c \cap B^c) = 1 - P(A \cup B) = 0.5$.
- $P(C^c | (A^c \cap B^c)) = \dfrac{P(A^c \cap B^c \cap C^c)}{P(A^c \cap B^c)} = \dfrac{0.3}{0.5} = 0.6$.

2. [20 points] Dilbert has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with $P(H) = p \neq \frac{1}{2}$. The probability that a coin chosen at random from his pocket will land Tails is $\frac{7}{12}$.

(a) [8 points] What is the value of p?

(b) [12 points] Dilbert picks two coins at random from his pocket, tosses each coin once, and observes a Head and a Tail. What is the conditional probability that both coins are fair?

$P(T) = P(T|\text{fair coin})P(\text{fair coin}) + P(T|\text{biased coin})P(\text{biased coin}) = \frac{1}{2}\cdot\frac{2}{3} + (1-p)\cdot\frac{1}{3} = \frac{7}{12} \Rightarrow p = \frac{1}{4}$.

$P(\text{one Head, one Tail}|\text{two fair coins}) = P(\{HT, TH\}) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$.

$P(\text{one Head, one Tail}|\text{one fair, one biased}) = P(\{\text{fair} = H, \text{biased} = T\}) + P(\{\text{biased} = H, \text{fair} = T\})$
$= \frac{1}{2}\cdot(1-p) + p\cdot\frac{1}{2} = \frac{1}{2}$ regardless of the value of p. Therefore, the theorem of total probability gives

$$\begin{aligned} P(\text{one Head, one Tail}) &= P(H,T|\text{two fair coins})P(\text{two fair coins}) \\ &\quad + P(H,T|\text{one fair, one biased})P(\text{one fair, one biased}) \\ &= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} == \frac{1}{6} + \frac{1}{3} = \frac{1}{2}, \end{aligned}$$

and the *conditional* probability of both coins being fair given that one Head and one Tail was observed is, by Bayes' formula,

$$P(\text{both coins fair}|\text{one Head, one Tail}) = \frac{P(H,T|\text{two fair coins})P(\text{two fair coins})}{P(\text{one Head, one Tail})} = \frac{1/6}{1/2} = \frac{1}{3}$$

the same as the unconditional probability! What does this tell you about the events {both coins fair} and {one Head, one Tail}?

3. [40 points] Two players A and B are playing a tennis game. Player A wins each point with probability p while player B wins with probability $q = 1 - p$. Each point may be regarded as an independent trial. Let D denote the event that A and B each win 3 of the first 6 points played. The score is then said to be *deuce*, meaning tied.

(**a**) [6 points] What is $P(D)$?

(**b**) [18 points] Once the score reaches deuce, a player must win *two more* points than the opponent in order to win the game. Given that the score is deuce, what is the (conditional) probability that A wins the next two points (and hence the game)? that player B wins the next two points (and hence the game)? that each wins one of the next two points so that the score is deuce again?

(**c**) [6 points] What is the conditional probability that A (ultimately) wins the game given that D occurred?

(**d**) [6 points] Let $\mathcal{X}$ denote the number of points played till the end of the game. What is the conditional pmf of $\mathcal{X}$ given D? Note that either A or B could have won the game.

(**e**) [4 points]What is the conditional expected value of $\mathcal{X}$ given that D occurred?

$P(D) = \binom{6}{3} p^3 q^3 = 20p^3q^3$.
$P(A \text{ wins two points}) = p^2, P(B \text{ wins two points}) = q^2, P(A, B \text{ win one point each}) = pq + qp = 2pq$. Note that the three probabilities add up to 1. The probability that A ultimately wins the game once deuce has been reached for the first time is

$$P(A \text{ ultimately wins}) = p^2 + (2pq)p^2 + (2pq)^2p^2 + (2pq)^3p^2 + \cdots = \frac{p^2}{1-2pq} = \frac{p^2}{p^2+q^2}.$$

Similarly, $P(B \text{ ultimately wins}) = \dfrac{p^2}{p^2+q^2}$.
Given that D has occurred, 6 points have been played already. The number of additional points played is $2, 4, 6, \ldots, n, \ldots$ and hence, conditioned on D, the random variable $\mathcal{X}$ takes on values $8, 10, 12, \ldots, n, \ldots$ with probabilities $(p^2+q^2), (p^2+q^2)(2pq), (p^2+q^2)(2pq)^2, \ldots, (p^2+q^2)(2pq)^{(n-8)/2}, \ldots$ Recalling that $2pq = 1 - (p^2+q^2)$, we note that conditioned on the occurrence of D, we can write $\mathcal{X} = 6 + 2\mathcal{Y}$ where $\mathcal{Y}$ is a *geometric* random variable with parameter (p^2+q^2).
$\mathsf{E}[\mathcal{X}|D] = \mathsf{E}[6 + 2\mathcal{Y}] = 6 + 2 \cdot \dfrac{1}{p^2+q^2} = 6 + \dfrac{2}{p^2+q^2}$. Alternatively, grind it out from the pmf found in part (d).

4. (**a**) [6 points] Find $\mathsf{E}[\mathcal{X}^2]$ for a Poisson random variable $\mathcal{X}$ with mean 4.

(**b**) [6 points] If $\mathcal{Y}$ is a geometric random variable with mean 4, what is $\mathsf{var}(2 + 3\mathcal{Y})$?

(**c**) [8 points] If $\mathcal{Z}$ denotes the number of occurrences of an event of probability p on 10 independent trials, what is the *conditional* expected value of $\mathcal{Z}$ given that the event occurred 4 times on the first six trials?

Since a Poisson random variable has mean and variance both equal to its parameter λ, and we are given that $\lambda = 4$ in this case, we get that $\mathsf{E}[\mathcal{X}^2] = \mathsf{var}(\mathcal{X}) + (\mathsf{E}[\mathcal{X}])^2 = 4^2 + 4 = 20$.
Since a geometric random variable has mean $1/p$ and variance $(1-p)/p^2$, and we are given that $1/p = 4$ in this case, we get that $\mathsf{var}(2 + 3\mathcal{Y}) = 3^2\mathsf{var}(\mathcal{Y}) = 9 \cdot \dfrac{3/4}{1/16} = 108$.
What happens on the last 4 trials is independent of what happens on the first 6. Thus, the number of occurrences of the event on the last 4 trials is a binomial random variable with parameters $(4, p)$, and hence expected value $4p$. This is true regardless of what happened on the first six trials. Therefore the conditional expectation of $\mathcal{Z}$ is $4 + 4p$.