1. [20 points] Let $A$, $B$, and $C$ denote three events defined on a sample space $\Omega$, and suppose that $P(A) = P(B) = 0.3, P(C) = 0.5$, and $P(A \cap B^c) = P(A^c \cap B \cap C) = 0.2$. Find the following probabilities: $P(A \cap B), P(A^c \cap B)$, $P((A \cup B \cap C)^c)$, and $P(C^c | (A \cap B^c))$.

The Karnaugh maps shown below are very useful in visualizing the problem.

We get that

- $P(A \cap B) = P(A) - P(A \cap B^c) = 0.3 - 0.2 = 0.1$. Note that $P(A \cup B) = 0.5$.
- $P(A^c \cap B) = P(B) - P(A \cap B) = 0.3 - 0.1 = 0.2$.
- $P((A \cup B) \cap C) = 1 - P((A \cup B) \cup C) = 1 - [P(A \cup B) + P(A^c \cap B \cap C)] = 1 - [0.5 + 0.2] = 0.3$.
- $P(C^c | (A \cap B^c)) = \frac{P(A^c \cap B^c \cap C^c)}{P(A^c \cap B^c)} = \frac{0.3}{0.5} = 0.6$.

2. [20 points] Dilbert has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with $P(H) = p \neq \frac{1}{2}$. The probability that a coin chosen at random from his pocket will land Tails is $\frac{7}{12}$.

(a) [8 points] What is the value of $p$?

(b) [12 points] Dilbert picks two coins at random from his pocket, tosses each coin once, and observes a Head and a Tail. What is the conditional probability that both coins are fair?

$P(T) = P(T | \text{fair coin})P(\text{fair coin}) + P(T | \text{biased coin})P(\text{biased coin}) = \frac{1}{2} \left( \frac{1}{2} \right) + \frac{1}{2} \left( 1 - p \right) = \frac{7}{12} \Rightarrow p = \frac{1}{4}$.

$P(\text{one Head, one Tail} | \text{two fair coins}) = P(\{HT, TH\}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$P(\text{one Head, one Tail} | \text{one fair, one biased}) = P(\{\text{fair = H, biased = T}\}) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

Regardless of the value of $p$. Therefore, the theorem of total probability gives

$P(\text{one Head, one Tail}) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

and the conditional probability of both coins being fair given that one Head and one Tail was observed is, by Bayes’ formula,

$P(\text{both coins fair} | \text{one Head, one Tail}) = \frac{P(H, T | \text{two fair coins})P(\text{two fair coins})}{P(\text{one Head, one Tail})} = \frac{1/6}{1/2} = \frac{1}{3}$

the same as the unconditional probability! What does this tell you about the events {both coins fair} and {one Head, one Tail}?
3. [40 points] Two players A and B are playing a tennis game. Player A wins each point with probability $p$ while player B wins with probability $q = 1 - p$. Each point may be regarded as an independent trial. Let $D$ denote the event that $A$ and $B$ each win 3 of the first 6 points played. The score is then said to be deuce, meaning tied.

(a) [6 points] What is $P(D)$?

(b) [18 points] Once the score reaches deuce, a player must win two more points than the opponent in order to win the game. Given that the score is deuce, what is the (conditional) probability that $A$ wins the next two points (and hence the game)? that player $B$ wins the next two points (and hence the game)? that each wins one of the next two points so that the score is deuce again?

(c) [6 points] What is the conditional probability that $A$ (ultimately) wins the game given that $D$ occurred?

(d) [6 points] Let $X$ denote the number of points played till the end of the game. What is the conditional pmf of $X$ given $D$? Note that either $A$ or $B$ could have won the game.

(e) [4 points] What is the conditional expected value of $X$ given that $D$ occurred?

$$P(D) = \binom{6}{3} p^3 q^3 = 20p^3 q^3.$$  

$P(A$ wins two points) = $p^2$, $P(B$ wins two points) = $q^2$, $P(A, B$ win one point each) = $pq + qp = 2pq$.  

Note that the three probabilities add up to 1. The probability that $A$ ultimately wins the game once deuce has been reached for the first time is

$$P(A \text{ ultimately wins}) = p^2 + (2pq)p^2 + (2pq)^2 p^2 + (2pq)^3 p^2 + \cdots = \frac{p^2}{1 - 2pq} = \frac{p^2}{p^2 + q^2}. $$

Similarly, $P(B$ ultimately wins) = $\frac{q^2}{p^2 + q^2}$.

Given that $D$ has occurred, 6 points have been played already. The number of additional points played is 2, 4, 6, . . . , $n$, . . . and hence, conditioned on $D$, the random variable $X$ takes on values 8, 10, 12, . . . , $n$, . . . with probabilities $(p^2 + q^2), (p^2 + q^2)(2pq), (p^2 + q^2)(2pq)^2, . . . , (p^2 + q^2)(2pq)^{(n-6)/2}, . . .$. Recalling that $2pq = 1 - (p^2 + q^2)$, we note that conditioned on the occurrence of $D$, we can write $X = 6 + 2Y$ where $Y$ is a geometric random variable with parameter $(p^2 + q^2)$.

$$E[X|D] = E[6 + 2Y] = 6 + 2 \cdot \frac{1}{p^2 + q^2} = 6 + \frac{2}{p^2 + q^2}. $$

Alternatively, grind it out from the pmf found in part (d).

4. (a) [6 points] Find $E[X^2]$ for a Poisson random variable $X$ with mean 4.

(b) [6 points] If $Y$ is a geometric random variable with mean 4, what is $\text{var}(2 + 3Y)$?

(c) [8 points] If $Z$ denotes the number of occurrences of an event of probability $p$ on 10 independent trials, what is the conditional expected value of $Z$ given that the event occurred 4 times on the first six trials?

Since a Poisson random variable has mean and variance both equal to its parameter $\lambda$, and we are given that $\lambda = 4$ in this case, we get that $E[X^2] = \text{var}(X) + (E[X])^2 = 4^2 + 4 = 20$.

Since a geometric random variable has mean $1/p$ and variance $(1-p)/p^2$, and we are given that $1/p = 4$ in this case, we get that $\text{var}(2 + 3Y) = 3^2 \text{var}(Y) = 9 \cdot \frac{3/4}{1/16} = 108$.

What happens on the last 4 trials is independent of what happens on the first 6. Thus, the number of occurrences of the event on the last 4 trials is a binomial random variable with parameters $(4, p)$, and hence expected value $4p$. This is true regardless of what happened on the first six trials. Therefore the conditional expectation of $Z$ is $4 + 4p$. 
