University of Illinois Spring 2005

ECE 413: Hour Exam I

1. [20 points] Let A, B, and C denote three events defined on a sample space Ω, and suppose that $P(A) = P(B) = 0.3, P(C) = 0.5$, and $P(A \cap B^c) = P(A^c \cap B^c \cap C) = 0.2$. Find the following probabilities: $P(A \cap B)$, $P(A^c \cap B)$, $P((A \cup B \cup C)^c)$, and $P(C^c|(A^c \cap B^c))$.

2. [20 points] Dilbert has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with $P(H) = p \neq \frac{1}{2}$. The probability that a coin chosen at random from his pocket will land Tails is $\frac{7}{12}$.

(**a**) [8 points] What is the value of p?

(**b**) [12 points] Dilbert picks two coins at random from his pocket, tosses each coin once, and observes a Head and a Tail. What is the conditional probability that both coins are fair?

3. [40 points] Two players A and B are playing a tennis game. Player A wins each point with probability p while player B wins with probability $q = 1 - p$. Each point may be regarded as an independent trial. Let D denote the event that A and B each win 3 of the first 6 points played. The score is then said to be *deuce*, meaning tied.

(**a**) [6 points] What is $P(D)$?

(**b**) [18 points] Once the score reaches deuce, a player must win *two more* points than the opponent in order to win the game. Given that the score is deuce, what is the (conditional) probability that A wins the next two points (and hence the game)? that player B wins the next two points (and hence the game)? that each wins one of the next two points so that the score is deuce again?

(**c**) [6 points] What is the conditional probability that A (ultimately) wins the game given that D occurred?

(**d**) [6 points] Let $\mathcal{X}$ denote the number of points played till the end of the game. What is the conditional pmf of $\mathcal{X}$ given D? Note that either A or B could have won the game.

(**e**) [4 points]What is the conditional expected value of $\mathcal{X}$ given that D occurred?

4. (**a**) [6 points] Find $\mathsf{E}[\mathcal{X}^2]$ for a Poisson random variable $\mathcal{X}$ with mean 4.

(**b**) [6 points] If $\mathcal{Y}$ is a geometric random variable with mean 4, what is $\mathsf{var}(2 + 3\mathcal{Y})$?

(**c**) [8 points] If $\mathcal{Z}$ denotes the number of occurrences of an event of probability p on 10 independent trials, what is the *conditional* expected value of $\mathcal{Z}$ given that the event occurred 4 times on the first six trials?