Assigned: Wednesday, February 19
Due: Wednesday, February 26
Reading: Yates and Goodman: Chapters 1, 2 and 3.6
Noncredit Exercises: (Do not turn these in) p. 20: Quiz 1.5; p. 27: Quiz 1.7; p. 79: Quiz 2.9; pp. 38-39: Problems 1.5.1 – 1.5.6, p. 86: 2.9.1 – 2.9.8

Problems:

1. A long message is divided into L packets of N bits each (including headers, address, CRC bits, tail, flags, etc. etc. etc.), and transmitted over a channel with bit error probability p. If the CRC detects that a packet is received in error, the packet transmission is repeated. *But*, if a packet has been transmitted a *total* of five times, and has not been received correctly even on the fifth try, it is not transmitted again, and is deemed to be lost.

(a) What is the probability that a packet is received correctly?

(b) Let $\mathbf{X}_i$ denote the number of times that the i-th packet is transmitted over the channel. What are the possible values of $\mathbf{X}_i$? What is the pmf of $\mathbf{X}_i$? (Be careful about $P\{\mathbf{X}_i = 5\}$!!).

(c) What is the average number of times that the i-th packet is transmitted? i.e., what is $E[\mathbf{X}_i]$?

(d) What is the probability that all L packets are received successfully = P{entire message is received successfully}?

2. This problem on conditional probability has three ***unrelated*** parts.

(a) If $P(A|B) = 0.3$, $P(A^c|B^c) = 0.4$, and $P(B) = 0.7$, find $P(A|B^c)$, $P(A)$, and $P(B|A)$.

(b) If $P(E) = 1/4$, $P(F|E) = 1/2$, and $P(E|F) = 1/3$, find $P(F)$.

(c) If $P(G) = P(H) = 2/3$, show that $P(G|H) \geq 1/2$.

3. Let $\mathbf{X}$ denote a *negative* binomial (or Pascal) random variable with parameters (r, p). Then, $\mathbf{X}$ counts the number of trials required to observe r successes where the probability of success on any trial is p. Given that $\mathbf{X} = n$, what is the conditional probability that the i-th trial resulted in a success? To avoid trivialities, assume that $n > r$ and also that $n > i$.

4. [“I’m leaving on a jet plane, Don’t know when I’ll be back again…”] We return to Problem 6 of Problem Set #4 where 105 passengers hold reservations for a 100–seat flight from Chicago to Champaign. Now suppose that 15 passengers are arriving in Chicago on a connecting flight that is on time with probability 1/3 and late with probability 2/3. If the connecting flight is on time, all 15 passengers show up for the flight to Champaign (no one stops off in a bar for a drink!); else they all are not there. The other 90 passengers each decide independently with probability 0.9, as before, to show up for the flight.

(a) Given that the connecting flight is on time, what is the (conditional) probability that all passengers who show up get seats? Repeat for the case when the connecting flight is late.

(b) Given that everyone who shows up gets a seat, what is the (conditional) probability that the connecting flight was on time?

5. An urn contains 6 red balls and 4 green balls. A ball is drawn at random from the urn, and then another ball is drawn at random without the first one being replaced in the urn.

(a) Let R_1 denote the event that the first ball is red and R_2 the event that the second ball is red. What is $P(R_1)$? What is $P(R_2|R_1)$? What is $P(R_2|R_1^c)$?

(b) Use the numbers found in part (a) to compute $P(R_2)$. Does the answer surprise you?

(c) Repeat parts (a) and (b) for an experiment in which after the first ball is drawn, *it is put back into the urn along with 3 more balls of the same color before the second ball is drawn.*

6. Monty Hall, the host of the TV game show “Let’s Make A Deal™”, shows you three curtains. One curtain conceals a valuable prize, while the other two conceal junk. All three curtains are equally likely to conceal the prize. He offers you the following “deal”: pick a curtain, and you can have whatever is behind it. When you pick a curtain, instead of giving you your just deserts, Monty (who knows where the prize is) opens one of the remaining curtains to show you that there is junk behind it, and offers the following “new, improved deal™” : you can either stick with your original choice, or switch to the remaining (unopened) curtain. Amidst the deafening roars of “Stand pat” and “Switch, you idiot” from the crowd, Monty points out that previously your chances of winning were 1/3. Now, since you know that the prize is behind one of the two unopened curtains, your chances of winning have increased to 1/2, and thus the new improved deal is indeed better. Use the theorem of total probability to determine

(a) the probability of winning if you always switch.

(b) the probability of winning if you would rather fight than switch.

(c) whether Monty is correct in asserting that if you choose randomly betweeen the two unopened curtains, you have a probability of winning of 1/2.

Note: Everybody knows that the rules of the game are that Monty always opens one of the two unchosen curtains and he always offers the “new improved deal,” i.e. he never opens a curtain to reveal the prize (saying “Oops, you lose; go back to your seat”)

7. At the County Fair, you see a man sitting at a table and rapidly rolling a pea between three walnut shells. “Step right up, me bucko, and try your luck! The hand is quicker than the eye!” he says, and hides the pea under one of the shells. You have no idea which shell is covering the pea, but you point to one shell at random and bet that the pea is under it. The man picks up one of the shells that you didn’t choose, and shows you that the pea is not underneath that shell. He asks if you would like to switch your bet to the other unchosen shell. Should you accept the offer? Why or why not? How does this game differ from the one analyzed in Problem 6?

More noncredit exercises:

1. [“Take me out to the ball game”] A baseball pitcher’s repertoire is limited to fastballs (event F), curveballs (event C), or sliders (event S). It is known that P(C) = 2P(F). The event H that the batter hits the ball has probabilities $P(H|F) = 2/5$, $P(H|C) = 1/4$, and $P(H|S) = 1/6$.
If P(H) = 1/4, what is P(C)?

2. Let $\mathbf{X}$ denote the number of Heads observed in 10 tosses of a fair coin.

(a) What kind of random variable is $\mathbf{X}$, and what is its average value?

(b) Find $P\{\mathbf{X} \leq 5 \mid \mathbf{X} \geq 4\}$.

(c) Given that $\mathbf{X} = 4$, what is the (conditional) probability that the 4th toss was a Head?

(d) You have a strong suspicion that the coin that was tossed is not a fair coin. Nonetheless, your friendly neighborhood bookmaker offers 2-to-1 odds if you want to bet that the 4th toss was a Head knowing only that the event $\{\mathbf{X} = 4\}$ has occurred. That is, if you bet \$1 that the 4th toss was a Head, then your wealth will increase by \$2 if the 4th toss did in fact show a Head, and your wealth will decrease by \$1 if the 4th toss resulted in a Tail. The bookie knows the outcome of the 4th toss (but cannot change it after you have placed your bet!) as well as the value of P(Head). Should you take the bet? Why or why not?

3. A denotes the event “rolling a seven” with two fair dice. When asked in class, several of you have responded that P(A) = 1/6 = 2/12. Let B_i denote the event that one of the dice is showing an i (the other may or may not be showing an i).

(a) Show that $P(A|B_3) = 2/11$, and that all the probabilities $P(A|B_i)$ have the same value.

(b) Since one of the events $B_1, B_2, B_3, B_4, B_5, B_6$ always occurs on a trial, we conclude that $\sum P(B_i) = 1$. It would thus seem from the theorem of total probability that $P(A) = \sum P(A|B_i)P(B_i) = (2/11)\sum P(B_i) = 2/11$. So, does P(A) equal 2/11 or 2/12? Explain where the error occurred in computing the other (wrong) answer.