University of Illinois | **Problem Set #2** | **ECE 313 Spring 2003**

Assigned: Wednesday, January 29
Due: Wednesday, February 5
Reading: Yates and Goodman: Sections 1.1-1.4, 1.8, 1.9, 2.1 and 2.2
Noncredit Exercises: (Do not turn these in) p. 11: Quiz 1.2; pp. 14-15: Quiz 1.3; p. 16: Quiz 1.4; pp. 30-31, Quiz 1.8; pp. 37-38: Problems 1.3.1–1.4.5; p. 41: 1.8.1–1.8.6 and 1.9.1.

Problems:

1. This problem continues the work begun in Problem 1 of the previous problem set. The results that you will (I hope!) obtain will be used repeatedly in the course.

(a) Suppose f(x) is a polynomial of degree $n > 0$. What is the n-th derivative of f(x)? What is the (n+1)-th derivative of f(x)? What is the (n+2)-th derivative of f(x)?

(b) Henceforth, let $f(x) = (1+x)^n$, $n > 0$.
True or False? If we multiply out the terms, we will get a polynomial of degree n.

(c) For $0 \le k \le n$, what is the k-th derivative of $(1+x)^n$, and what is the k-th term of the Taylor series around $x = 0$ (a.k.a. MacLaurin series) for $(1+x)^n$?

(d) Repeat part (e) for $k > n$.

(e) True or False? The MacLaurin series for $(1+x)^n$ contains terms of degree greater than n.
Congratulations! You have just re-discovered the binomial theorem for positive exponents.

Now suppose that $f(x) = (1-x)^{-n} = \dfrac{1}{(1-x)^n}$, where, as before, we assume that $n > 0$.

(f) Does the MacLaurin series for $(1-x)^{-n}$ contain terms of degree greater than n? If so, what is the (n+1)-th term in the MacLaurin series? If not, what is the n-th term in the MacLaurin series?

(g) Use the results of part (f) to write down the MacLaurin series for $(1-x)^{-1}$ and $(1-x)^{-2}$.
These two results and the one in parts (c)-(e) are used so often in the course that it is best to memorize them!

(h) I hope that you found that $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ which gives the curious result that $-1 = \infty$ on setting $x = 2$ on both sides. Is this correct? If not, what is wrong?

2. We noted that the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in the MacLaurin series (or binomial theorem) expansion of $(1+x)^n$, and it also denotes the number of subsets of size k from a set Ω of n objects.

(a) Use the binomial theorem to expand $(1-x)^n$. DO NOT use summation (Σ) notation: I want to see the first four terms explicitly listed, and I also want to know the coefficient of x^n.

(b) Compare the result of part (a) to the expansion of $(1+x)^n$. Some powers of x have positive signs in both expansions. Which powers are these? Which powers have opposite signs?

(c) We saw on Problem Set #1 that the MacLaurin series for $(1+x)^n + (1-x)^n$ has even powers of x only. So, what *is* the coefficient of x^{2k} in the series?

(d) Let S_{even} denote the *collection* of subsets of size k where k, $0 \le k \le n$, is an even number. Note that zero is an even number and hence the empty set is always a member of the collection S_{even}. However, Ω is not a member of S_{even} whenever n is an odd number. Set $x = 1$ in the result of part (c) and thus show that $|S_{even}| = 2^{n-1}$.

(e) Explain why the result of part (d) implies that S_{odd}, the collection of subsets containing an odd number of objects, also is a collection of 2^{n-1} subsets.

3.(a) An ice-cream manufacturer makes unflavored ice-cream and then creates "specialty flavors" by blending one or more of the five essences: vanilla, chocolate, fudge, mint, and almond. Thus, vanilla almond fudge ice-cream is created by blending in the three essences vanilla, almond, and fudge into the unflavored ice-cream. How many specialty flavors can the manufacturer create? **Optional non-credit exercise:** Identify the manufacturer!

(b) Events A and B are events defined on a sample space. If the probability that at least one of the two events occurred is 0.6, and the probability that at least one of the events did not occur is 0.8, what is the probability that *exactly one* of the events A and B did not occur?

4. A certain town has three newspapers A, B, and C. The proportions of townspeople that read these newspapers are as follows:
A: 10%, B: 30%, C: 5%; A and B: 8%, A and C: 2 %, B and C: 4%, and 1% read all three newspapers.

(a) What percentage of people read only one newspaper?
(b) What percentage of people read at least two newspapers?
(c) If A and C are morning papers and B is an evening paper, what percentage of people read at least one morning paper as well as the evening paper?
(d) How many people do not read any newspapers?

5. An experiment consists of observing the contents of an ten-bit shift register. Assume that all 2^{10} = 1024 bytes are equally likely to be the contents of the shift register.

(a) Let A denote the event that the least significant bit in the shift register is a 1. What is P(A)?
(b) Let B denote the event that the shift register contains 4 1's and 6 0's. What is P(B)?
(c) What is P(A B) ? What is P(A ∩ B)? What is the probability that exactly one of the events A and B occurs, i.e. what is P(A B)?

6. The manufacturer of a cereal tests samples from the production line to see if the samples snap, crackle, and pop as advertised. Let A, B, and C denote respectively the events that the sample **does not** snap, **does not** crackle, and **does not** pop. The manufacturer's tests show that P(A) = P(C) = 0.3, P(B) = 0.2, P(AB BC AC) = 0.3, P(ABC) = 0.05, P(AB) = 0.1, and P(AC) = 2P(BC).

(a) Sketch the sample space and indicate on it the events A, B, and C.
(b) What is the probability that the cereal snaps, crackles, and pops ?
(c) Cereal that fails exactly one test is sold to discount supermarket chains at lower prices to be marketed under the brand names Soggies, Blecchies, and Mushies. What is the probability of the sample failing the snap test **only**? the crackle test **only**? the pop test **only**?