1.(a) There are $\binom{6}{3} = \frac{6\times5\times4}{1\times2\times3} = 20$ short lists of which $\binom{5}{2} = \frac{5\times4}{1\times2} = 10$ include Chuck, and thus Chuck has probability 10/20 = 1/2 of being on the short list. By symmetry, everyone has probability 1/2 of being on the short list.

(b) There is only one short list {Andy, Chuck, Eddie} that has no women on it. Hence, the probability is 19/20 that there is at least one woman on the short list.

(c) Chuck is selected as monarch with conditional probability 1/3 if he is on one of the three short lists that include him and two of Andy, Eddie and Fergie, and he is selected as monarch with probability 1/2 if he is on of the three short lists that include him and Di and one of Andy, Eddie and Fergie. Hence, from the theorem of total probability we get that P{Chuck selected as monarch} = $\frac{1}{3}\times\frac{3}{20} + \frac{1}{2}\times\frac{3}{20} = \frac{1}{8}$. More directly, since Beth is on the short list with probability 1/2, she is selected as monarch with probability 1/2 also. By symmetry, the other 4 nonDians are equally likely to be selected if Beth is not, and thus are selected with probability 1/8 each.

(d) $P\{\text{Di on list} \mid \text{Chuck selected}\} = \frac{P\{\text{Chuck selected and Di on list}\}}{P\{\text{Chuck selected}\}}$

$= \frac{P\{\text{Chuck selected} \mid \text{C, D and one of A, E, F on list}\}P\{\text{C, D and one of A, E, F on list}\}}{P\{\text{Chuck selected}\}}$

$= \frac{(1/2)(3/20)}{(1/8)} = \frac{3}{5}$.

(e) The other two women must be Di and Fergie (not Beth, since Chuck was selected), and hence from part (d) we get that P{two women on list | Chuck selected} = 1/5.

2. P{HHH} = P{TTT} = 1/8. Thus, a decision is made with probability 3/4 and re-tossing is required with probability 1/4. The number of rounds $\mathbf{X}$ is thus a geometric random variable with parameter p = 3/4. Note (from symmetry) that Groucho, Chico and Harpo have equal probability 1/3 of winning the game.

(a) P{at least 3 rounds required} = $P\{\mathbf{X} > 2\} = q^2 = (1/4)^2 = 1/16$.

(b) $P\{\mathbf{X} \text{ is even}\} = pq + pq^3 + pq^5 + \ldots = \frac{pq}{1-q^2} = \frac{q}{1+q} = \frac{1/4}{1+1/4} = \frac{1}{5}$.

(c) $P\{\mathbf{X} > 2 \mid \mathbf{X} \text{ is even}\} = \frac{P\{\mathbf{X} > 2 \text{ and } \mathbf{X} \text{ is even}\}}{P\{\mathbf{X} \text{ is even}\}} = \frac{pq^3 + pq^5 + \ldots}{1/5} = \frac{1/5 - 3/16}{1/5} = 1 - \frac{15}{16} = \frac{1}{16}$.

(d) Now the outcomes are {HHH, THH, HHT, THT} so that a decision is still made with probability 3/4 and re-tossing is required with probability 1/4. Each player still has the same probability 1/3 of winning.

(e) Now the outcomes are {HHT, THT} so that either Chico or Harpo wins (with probability 1/2) on the very first round. So, Harpo actually helps Chico by his strategy. Note that if Harpo had also substituted a two-headed coin, then Groucho wins with probability 1, though not necessarily on the first round!

3. Since the area under the pdf must be 1, we readily get that a + b = 2.

(a) The variance is a measure of the spread of the mass, and therefore is smallest when either a = 0 and b = 2 or when a = 2 and b = 0. In either case, the pdf is a uniform density on an interval of length 1/2, and hence the minimum variance is $(1/2)^2/12 = 1/48$.

More analytically, $\int_0^{1/2} u\,du = \frac{1}{8}$ and $\int_{1/2}^{1} u\,du = \frac{3}{8}$ from which we get that $E[\mathbf{X}] = \frac{a + 3b}{8} = \frac{3 - a}{4}$.

Similarly, $\int_0^{1/2} u^2\,du = \frac{1}{24}$ and $\int_{1/2}^{1} u^2\,du = \frac{7}{24}$ from which we get that $E[\mathbf{X}^2] = \frac{a + 7b}{24} = \frac{7 - 3a}{12}$.

Hence, $\text{var}(\mathbf{X}) = E[\mathbf{X}^2] - (E[\mathbf{X}])^2 = \frac{7 - 3a}{12} - \frac{9 - 6a + a^2}{16} = \frac{1 + 6a - 3a^2}{48}$. As a function of a, this quadratic has minimum value 1/48 on the interval [0, 2] , and the minimum value occurs at a = 0 and a = 2. Those who set the derivative to 0 and got the extremum at a = 1 actually found the maximum variance 1/12, not the minimum variance! The second derivative test, if tried, would have revealed the error.

(b) $F_{\mathbf{X}}(3/4) = \frac{a}{2} + \frac{b}{4} = \frac{a + 2}{4}$.

(c) If $E[\mathbf{X}] = \frac{3 - a}{4} = \frac{5}{8}$, we readily get $a = \frac{1}{2}$ and hence $F_{\mathbf{X}}(3/4) = \frac{a + 2}{4} = \frac{5}{8}$.

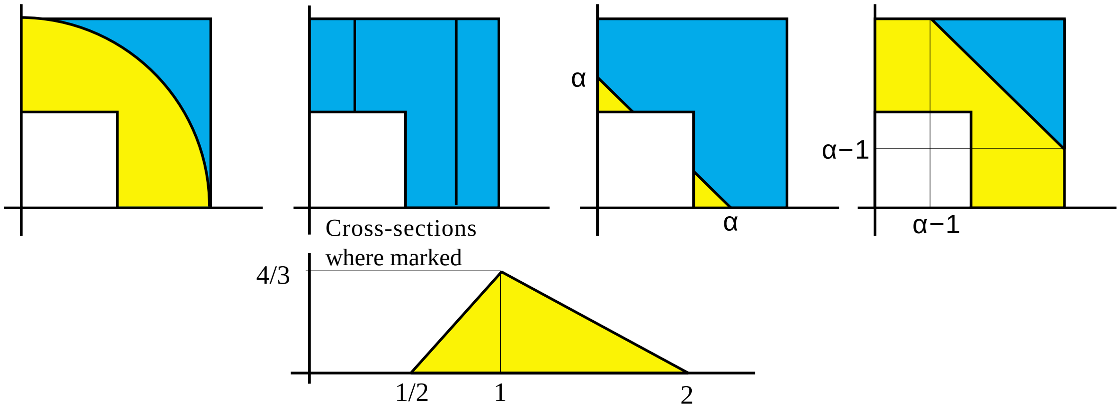
4.(a) $\mathbf{Y} = \exp(\mathbf{X})$ takes on values in the range $(1,\infty)$ as $\mathbf{X}$ varies between $(0,\infty)$.

For $v > 1$, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \le v\} = P\{\exp(\mathbf{X}) \le v\} = P\{\mathbf{X} \le \ln v\} = 1 - \exp(-\lambda(\ln v)) = 1 - v^{-\lambda}$. Note $\lambda > 0$.

Hence, $f_{\mathbf{Y}}(v) = \begin{cases} \lambda v^{-\lambda-1}, & v > 1, \\ 0, & v \le 1. \end{cases}$

5.(a) $P\{\mathbf{X}^2+\mathbf{Y}^2 < 1\} = P\{(\mathbf{X}, \mathbf{Y})$ shaded region shown in left-hand figure below$\} = \left[\frac{\pi}{4} - \frac{1}{4}\right]\bullet\frac{4}{3} = \frac{\pi-1}{3}$

(b) $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ takes on values between 1/2 and 2.



(c) For $1/2 \le \alpha \le 1$, $F_{\mathbf{Z}}(\alpha) = P\{\mathbf{Z} \le \alpha\} = P\{(\mathbf{X}, \mathbf{Y})$ shaded region shown in third figure above$\}$

$= \left[\left(\alpha - \frac{1}{2}\right)^2\right]\bullet\frac{4}{3} = \frac{4\alpha^2 - 4\alpha + 1}{3}$ $f_{\mathbf{Z}}(\alpha) = \frac{8\alpha - 4}{3}$ for $1/2 \le \alpha \le 1$.

For $1 \le \alpha \le 2$, $F_{\mathbf{Z}}(\alpha) = P\{\mathbf{Z} \le \alpha\} = P\{(\mathbf{X}, \mathbf{Y})$ shaded region shown in right-hand figure above$\}$

$= 1 - \frac{1}{2}\left[(2 - \alpha)^2\right]\bullet\frac{4}{3} = 1 - \frac{8 - 8\alpha + 2\alpha^2}{3}$ $f_{\mathbf{Z}}(\alpha) = \frac{8 - 4\alpha}{3}$ for $1 \le \alpha \le 2$.

The pdf of $\mathbf{Z}$ is thus as shown on the figure on the second row.

(d) The pdf is clearly nonnegative and the area under the pdf is easily found to be 1.

6. The joint pdf has value $4/\pi$ on the quadrant of the unit disc. Changing to polar coordinates simplifies the integrals in this problem.

(a) $E[\mathbf{X}] = \int\int u\, f(u,v)\, du\, dv = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r\cos\theta\,(4/\pi)\, r\, d\theta\, dr = \int_{r=0}^{1} r\,(4/\pi)\, r\, dr = 4/3\pi$.

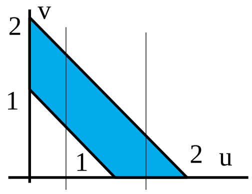
(b) Using the hint, $E[\mathbf{X}^2+\mathbf{Y}^2] = \int\int (u^2 + v^2) f(u,v)\, du\, dv = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^2(\cos^2\theta + \sin^2\theta)(4/\pi)\, r\, d\theta\, dr$

$= \int_{r=0}^{1} 2r^3\, dr = 1/2$. But, from symmetry, $E[\mathbf{X}^2] = E[\mathbf{Y}^2]$ and hence both have value 1/4. Otherwise, we have $E[\mathbf{X}^2] = \int\int u^2 f(u,v)\, du\, dv = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^2\cos^2\theta\,(4/\pi)\, r\, d\theta\, dr$. Integrating $\cos^2\theta$ by parts (*two* integrations by parts are needed!) gives $[\theta/2 + (1/4)\bullet\sin 2\theta]\Big|_0^{\pi/2} = \pi/4$, and thus $E[\mathbf{X}^2] = 1/4$. Either way, we get $\mathrm{var}(\mathbf{X}) = E[\mathbf{X}^2] - (E[\mathbf{X}])^2 = \frac{1}{4} - \frac{16}{9\pi^2} = \frac{9\pi^2 - 64}{36\pi^2}$.

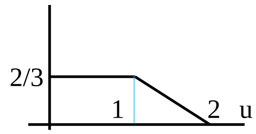
(c) $E[\mathbf{XY}] = \int\int u\, v\, f(u,v)\, du\, dv = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^2\cos\theta\sin\theta\,(4/\pi)\, r\, d\theta\, dr = \int_{r=0}^{1} r^3 \sin^2\theta\Big|_0^{\pi/2}(2/\pi)\, dr = 1/2\pi$.

Hence, $\mathrm{cov}(\mathbf{X}, \mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = \frac{1}{2\pi} - \frac{16}{9\pi^2} = \frac{9\pi - 32}{18\pi^2} < 0$. This is quite reasonable: after all, the *linear* MMSE estimator ought to be estimating small values for $\mathbf{Y}$ if $\mathbf{X}$ is large, and large values for $\mathbf{Y}$ if $\mathbf{X}$ is small, which means the slope of the line must be negative.

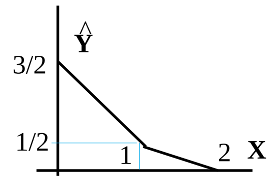
7. The joint pdf has value 2/3 on the region shown in the left-hand figure below.



Problem 7: joint pdf



Problem 7: pdf of X



MMSE estimator

(a) $f_{\mathbf{X}}(u)$, the value of the marginal pdf of **X** at u, equals the cross-sectional area of the pdf surface at u. Hence, we get that $f_{\mathbf{X}}(u)$ has constant value 2/3 for 0 < u < 1, and decreases to 0 as u increases from 1 to 2, as showb in the middle figure above. More formally,

$$f_{\mathbf{X}}(u) = \begin{cases} 2/3 & \text{for } 0 < u < 1, \\ (2/3)\bullet(2-u) & \text{for } 1 \le u < 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that the area under the pdf is 1.

(b) The MMSE estimator for **Y** given **X** is the mean of the conditional pdf of **Y** given the value of **X**. Also, the conditional pdf is simply the cross-section of the joint pdf surface "normalized" to have area 1.
It is easily seen that if **X** = a where 0 < a < 1, then the conditional pdf of **Y** is uniform on (1–a, 2–a) and hence has mean (3/2) – a which varies from 3/2 at a = 0 to 1/2 at a = 1, while if 1 ≤ a < 2, then the conditional pdf of **Y** is uniform on (0, 2–a) and thus has mean 1 – a/2 which varies from 1/2 at a = 1 to 0 at a = 2. Thus, the MMSE estimator $\hat{\mathbf{Y}}$ is (3/2)–**X** if 0 < **X** < 1, and 1 – **X**/2 if 1 ≤ **X** < 2, as illustrated in the right-hand figure above. Note that the function is *piecewise* linear, and thus is different from the linear MMSE estimator.