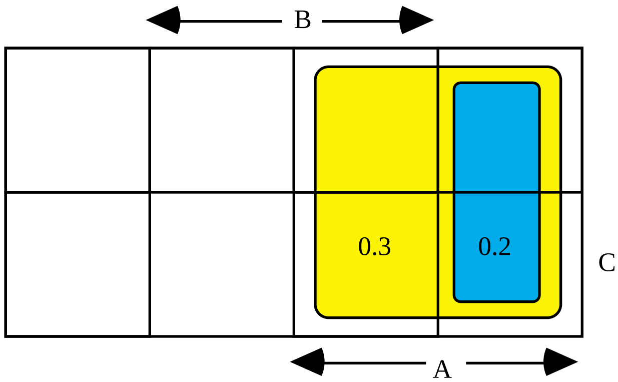
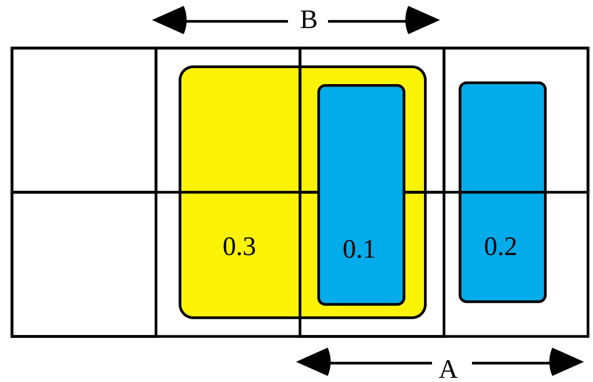
University of Illinois — **Solutions to First Hour Exam** — **ECE 313 Spring 2003**

Two different versions of the exam, differing in minor ways, were used.
Solutions are given in detail for one version.

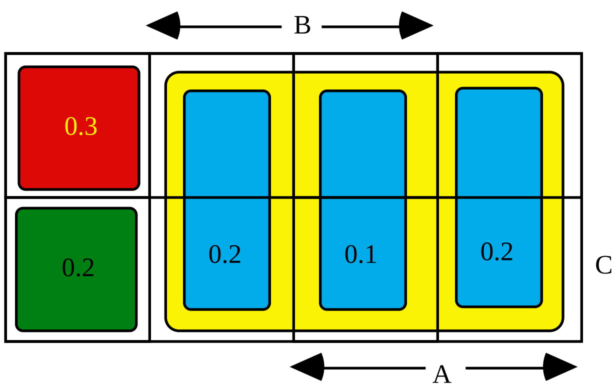
1. Given P(A) = 0.3, P(B) = 0.3, P(C) = 0.5, $P(A \cap B^c) = P(A^c \cap B^c \cap C) = 0.2$.

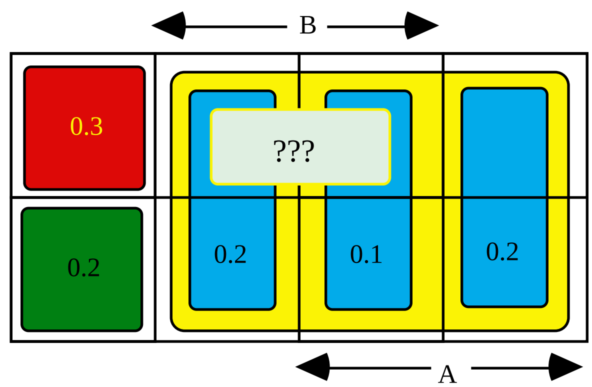
(a) A Karnaugh map/Venn diagram is very useful in solving problems such as these.





Since A is the union of the disjoint sets $A \cap B$ and $A \cap B^c$,
$P(A) = P(A \cap B) + P(A \cap B^c)$ $\quad P(A \cap B) = P(A) - P(A \cap B^c) = 0.3 - 0.2 = 0.1$.
Similarly, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A) + P(A \cap B^c) = 0.3 - 0.3 + 0.2 = 0.2$.





Next, $P(A \quad B \quad C) = P(A \quad B\ ) + P((A \quad B\ )^c \cap C) = P(A \quad B) + P(A^c \cap B^c \cap C)$ (why?)
$= P(A) + P(B) - P(A \cap B) + P(A^c \cap B^c \cap C) = 0.3 + 0.3 - 0.1 + 0.2 = 0.7$,
and, of course, $P((A \quad B \quad C)^c) = 1 - P(A \quad B \quad C) = 1 - 0.7 = 0.3$. $P(B \cap C^c)$ cannot be determined.
Finally, $P(C^c \mid (A^c \cap B^c)) = P(A^c \cap B^c \cap C^c)/P(A^c \cap B^c)$
$= P(A^c \cap B^c \cap C\ )/[P(A^c \cap B^c \cap C^c) + P(A^c \cap B^c \cap C)]$
$= P((A \quad B \quad C)^c)/[P((A \quad B \quad C)^c) + P(A^c \cap B^c \cap C)] = 0.3/(0.3 + 0.2) = 3/5$.
Note that the value of P(C) is not used anywhere except in deciding that $P(B \cap C^c)$ cannot be determined.
The alternative version swapped events A and C, and thus the above Karnaugh maps are still valid upon swapping A and C. Now, $P(A \cap B)$ cannot be determined, but $P(B \cap C) = 0.1$, $P((A \quad B \quad C)^c) = 0.3$, $P(B \cap C^c) = 0.2$, while $P(A^c \cap B^c)$ cannot be determined, and hence neither can $P(C^c \mid (A^c \cap B^c))$.

2.(a) **X** is a binomial random variable with parameters (10, 1/2), and hence E[**X**] = np = 10•(1/2) = 5.

(b) The number of heads on the first 6 tosses is a binomial random variable **Y** with parameters (6, 1/2).

$$P(A) = P\{\mathbf{Y} = 4\} = \binom{6}{4}\cdot\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 = \binom{6}{2}\cdot\left(\frac{1}{2}\right)^6 = \frac{15}{64}.$$

(c) If the event A has occurred, there have been 2 tails on the first six tosses. The number of tails on the remaining four tosses is independent of A, and has average value 4•(1/2) = 2. Hence, E[**X**|A] = 2+2 = 4.

(d) $P(B) = P\{2 \text{ heads on 6 tosses}\} = \binom{6}{2}\cdot\left(\frac{1}{2}\right)^6 = \frac{15}{64} = P(A)$. Now, divide the 10 tosses into three groups: the first 4, the middle 2, and the last 4 tosses. If the event AB occurs, then the numbers of heads in these three groups must be 4, 0, 2 respectively, or 3, 1, 1, respectively, or 2, 2, 0 respectively. Hence, we get that $P(AB) = \left[\binom{4}{4}\cdot\binom{2}{0}\cdot\binom{4}{2} + \binom{4}{3}\cdot\binom{2}{1}\cdot\binom{4}{1} + \binom{4}{2}\cdot\binom{2}{2}\cdot\binom{4}{0}\right]\cdot 2^{-10} = \frac{44}{1024} = \frac{11}{256}$.

Thus, $P(A|B) = \frac{P(AB)}{P(B)} = \frac{11}{256} \times \frac{64}{15} = \frac{11}{60}$.

The alternative version interchanged A and B, which changes the answer to part (c) to $E[\mathbf{X}|A] = 4+2 = 6$.

3.(a) There are $\binom{5}{2} = \frac{5 \times 4}{1 \times 2} = 10$ possible pairs of letters, consisting of the pair {I, I}, 6 pairs of the form {I, X} where X is one of {M, M, A}, 2 pairs {M, A}, and the pair {M, M}. The conditional probability that the sign still seems to read MIAMI is 1, 1/4, 1/8, and 1/4 respectively for these pairs falling down. The theorem of total probability thus gives

$$P\{\text{sign still seems to read MIAMI}\} = \left[1 \times \frac{1}{10} + \frac{1}{4} \times \frac{6}{10} + \frac{1}{8} \times \frac{2}{10} + \frac{1}{4} \times \frac{1}{10}\right] = \frac{12}{40} = \frac{3}{10}.$$

(b) From Bayes' formula, we get that P{2 M's fell down |sign still seems to read MIAMI}

$$= \frac{P\{\text{sign still seems to read MIAMI }|\text{2 M's fell down}\}P\{\text{2 M's fell down}\}}{P\{\text{sign still seems to read MIAMI}\}} = \frac{1/40}{12/40} = \frac{1}{12}.$$

The alternative version asked for the conditional probability that the two I's fell down, which is $\frac{4/40}{12/40} = \frac{1}{3}$.

4. $E[\mathbf{X}] = \text{var}(\mathbf{X}) = \lambda$.

$E[(\mathbf{X} - 2)^2] = E[\mathbf{X}^2] - 4\cdot E[\mathbf{X}] + 4 = \text{var}(\mathbf{X}) + (E[\mathbf{X}])^2 - 4\cdot E[\mathbf{X}] + 4 = \lambda + \lambda^2 - 4\cdot\lambda + 4 = \lambda^2 - 3\cdot\lambda + 4$.

Alternatively, the theorem of parallel axes gives $E[(\mathbf{X} - 2)^2] = E[(\mathbf{X} - 2)^2] + (\lambda - 2)^2 = \lambda + (\lambda - 2)^2$.

The numerical answers are 4 or 8 according as $\lambda = 3$ or 4.