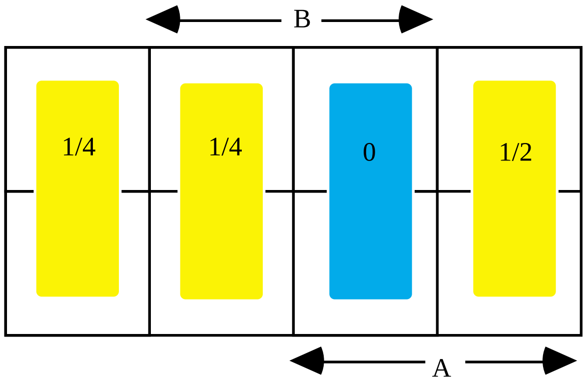
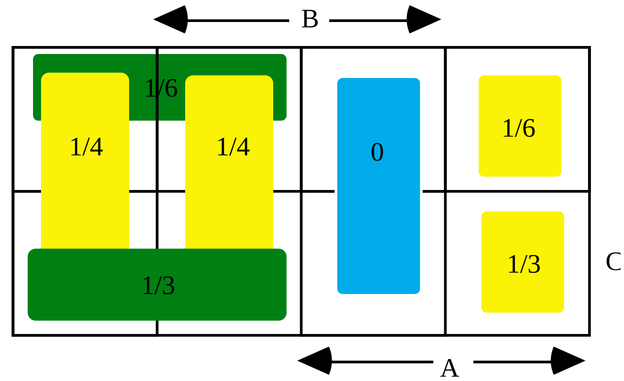
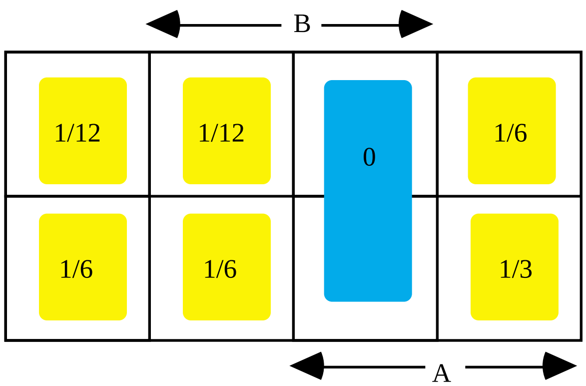
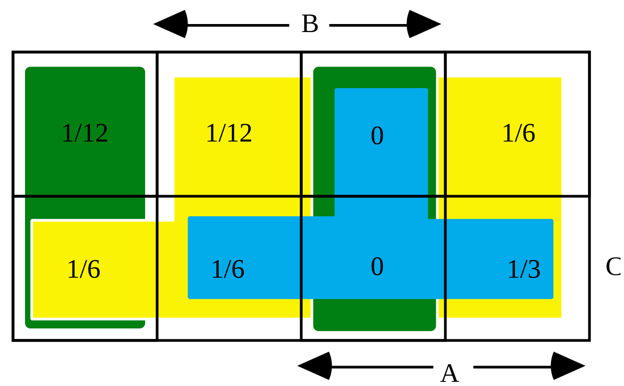
**1.** This problem is most easily solved with a Karnaugh map. From the information about disjoint events A and B, we readily obtain the map on the left below. Those who refuse to draw Karnaugh maps and visualize the problem* should work it out the hard way as P(AB) = 0 implies $P(AB^c) = P(A) – P(AB) = 1/2$ and $P(A^cB) = P(B) – P(AB) = 1/4$, and hence $P(A^cB^c) = P(A^c) – P(A^cB) = 1/2 – 1/4 = 1/4$ also. Reverting back to diagrams, we see that since A and C are independent, we get that P(AC) = P(A)P(C) = (1/2)(2/3) = 1/3, which gives the diagram on the right.



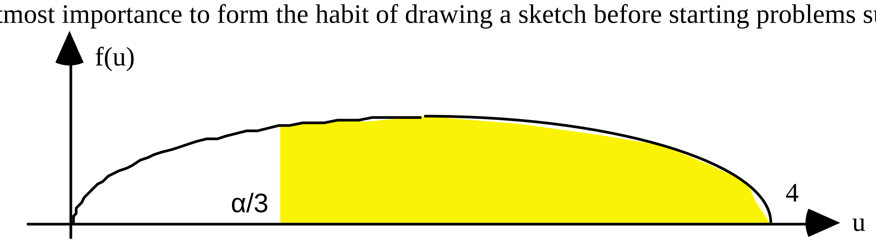

Next, note that since P(B|C) = 1/4, we get that P(BC) = P(B|C)P(C) = (1/4)(2/3) = 1/6 which gives the left-hand diagram below. Finally, the answers can be obtained as shown on the diagram on the right. We get that P(A  B  C) = 11/12, P(AB  BC  CA) = 1/2, and $P(C^c \mid A \; B^c) = (1/12)/3/12 = 1/3$.




Two slightly different versions of Problems 2-4 were used on the exam. Generic solutions are provided below.

**2.** It is of the utmost importance to form the habit of drawing a sketch before starting problems such as these.




The problem asked for $P\{3\mathbf{X}^2 < (15 + \alpha)\mathbf{X} – 5\alpha\}$ where $\alpha = 2$ or 4 depending on the version of the exam. Now, $\{3\mathbf{X}^2 < (15 + \alpha)\mathbf{X} – 5\alpha\} = \{3\mathbf{X}^2 – (15 + \alpha)\mathbf{X} + 5\alpha < 0\} = \{(3\mathbf{X} – \alpha)(\mathbf{X} – 5) < 0\}$, that is, **either** $\{(3\mathbf{X} – \alpha) < 0$ **and** $(\mathbf{X} – 5) > 0\}$ **or** $\{(3\mathbf{X} – \alpha) > 0$ **and** $(\mathbf{X} – 5) < 0\}$. The first condition is equivalent to $\mathbf{X} < \alpha/3$ **and** $\mathbf{X} > 5$ which obviously cannot be satisfied. The second condition is equivalent to $\{\alpha/3 < \mathbf{X} < 5\}$ and the probability of this event is the area under the pdf between $\alpha/3$ and 5, that is, the

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* “There are none so blind as those who will not see…”

shaded area ahown in the figure. We get $\int_{a/3}^{5} f(u)\,du = \int_{a/3}^{4} (\pi/8)\sin(\pi u/4)\,du$ since the pdf is zero between 4 and 5. We get $P\{\{3\mathbf{X}^2 < (15 + \alpha)\mathbf{X} - 5\alpha\} = -\frac{1}{2}\cos(\pi u/4)\Big|_{a/3}^{4} = \frac{1}{2}[1 + \cos(\pi\alpha/12)] = \frac{2 + \sqrt{3}}{4}$ or $\frac{3}{4}$ depending on the version of the exam.

**3.(a)** $P\{|\mathbf{X}| < 8\} = P\{-8 < \mathbf{X} < 8\} = \Phi((8 - \mu)/\sigma) - \Phi((-8 - \mu)/\sigma) = \Phi(0.1\bullet(8 - \mu)) - [1 - \Phi(0.1\bullet(8 + \mu))]$
$= \Phi(0.6) + \Phi(1) - 1 = 0.5670\ldots$ or $\Phi(0.4) + \Phi(1.2) - 1 = 0.5403\ldots$ depending on the version of the exam. A common error was using $\sigma = 100$ rather than $\sigma = 10$ in the calculations.

**(b)** $E[(\mathbf{X} - \alpha)^2] = E[\mathbf{X}^2 - 2\alpha\mathbf{X} + \alpha^2] = E[\mathbf{X}^2] - E[2\alpha\mathbf{X}] + E[\alpha^2] = E[\mathbf{X}^2] - 2\alpha E[\mathbf{X}] + E[\alpha^2]$
$= \mathrm{var}(\mathbf{X}) + \mu^2 - 2\alpha\mu + \alpha^2 = \mathrm{var}(\mathbf{X}) + (\mu - \alpha)^2 = 100 + 4 = 104$ for both versions of the exam.

**(c)** $\mathbf{Y} = (\mathbf{X} - \mu)^2$ takes on values $\geq 0$ only. Hence, for any $v \geq 0$,
$F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \leq v\} = P\{(\mathbf{X} - \mu)^2 \leq v\} = P\{-\sqrt{v} \leq \mathbf{X} - \mu \leq \sqrt{v}\} = P\{\mu - \sqrt{v} \leq \mathbf{X} \leq \mu + \sqrt{v}\}$
$= \Phi((\mu + \sqrt{v} - \mu)/\sigma) - \Phi((\mu - \sqrt{v} - \mu)/\sigma) = \Phi(\sqrt{v}/\sigma) - \Phi(\sqrt{v}/\sigma)$.
Now, the derivative of $\Phi(x)$ is $\varphi(x) = (1/\sqrt{2\pi})\bullet\exp(-x^2/2)$, and hence by the chain rule,
$f_{\mathbf{Y}}(v) = \frac{d}{dv}F_{\mathbf{Y}}(v) = \frac{d}{dv}[\Phi(\sqrt{v}/\sigma) - \Phi(\sqrt{v}/\sigma)] = (1/2\sigma)(1/\sqrt{2\pi v})\bullet\exp(-v/2\sigma^2) + (1/2\sigma)(1/\sqrt{2\pi v})\bullet\exp(-v/2\sigma^2)$
$= (1/\sigma\sqrt{2\pi v})\bullet\exp(-v/2\sigma^2) = (0.1/\sqrt{2\pi v})\bullet\exp(-0.005v)$for $v \geq 0$, and $f_{\mathbf{Y}}(v) = 0$ for $v < 0$. This is a gamma pdf with parameters $(1/2, 1/2\sigma^2) = (0.5, 0.005)$ as shown on e.g. Slide 31 of Powerpoint Lecture #27. Last week in class, we derived this result for the case $\sigma = 1$.

**4.** Let $\mu$ denote the arrival rate of the process. Then, both $\mathbf{N}(0,T]$ and $\mathbf{N}(0.5T, 1.5T]$ are Poisson random variables with parameter $\mu T$.

**(a)** Hence, $P(A) = P\{\mathbf{N}(0,T] = 1\} = \mu T\bullet\exp(-\mu T)$ and $P(B) = P\{\mathbf{N}(0.5T, 1.5T] = 0\} = \exp(-\mu T)$.

**(b)** $P(AB) = P\{\mathbf{N}(0,T] = 1, \mathbf{N}(0.5T, 1.5T] = 0\} = P\{\mathbf{N}(0,0.5T] = 1, \mathbf{N}(0.5T, 1.5T] = 0\}$ since the single arrival must have occurred during $(0, 0.5T]$. But, $\mathbf{N}(0,0.5T]$ and $\mathbf{N}(0.5T, 1.5T]$ are independent random variables because the time intervals are disjoint. Hence,
$P(AB) = P\{\mathbf{N}(0,0.5T] = 1, \mathbf{N}(0.5T, 1.5T] = 0\} = P\{\mathbf{N}(0,0.5T] = 1\}\bullet P\mathbf{N}(0.5T, 1.5T] = 0\}$
$= (\mu T/2)\bullet\exp(-\mu T/2)\bullet\exp(-\mu T) = (\mu T/2)\bullet\exp(-3\mu T/2)$, and $P(B|A) = P(AB)/P(A) = (1/2)\bullet\exp(-\mu T/2)$.