University of Illinois — **Problem Set #13** — **ECE 313**

 — **Spring 2002**

Assigned: Wednesday, April 17, 2002
Due: Wednesday, April 24, 2002
Reading: Ross, Chapter 6 and Chapter 7
Noncredit Exercises: Ross, Chapter 6: Problems 26, 28-30, 41-43, 51, 54;
Theoretical Exercises: 8, 14, 22, 23, 33;
Chapter 7: Problems 1, 16, 26, 29, 34, 36; Theoretical Exercises: 1, 2, 17, 22, 23, 40

Problems:

1. Let ($\mathbf{X}$, $\mathbf{Y}$) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} C\sqrt{1-u^2-v^2}, & u^2+v^2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$

(a) What is the value of C?

(b) Find $P\{\mathbf{X}^2+\mathbf{Y}^2 < 0.25\}$.

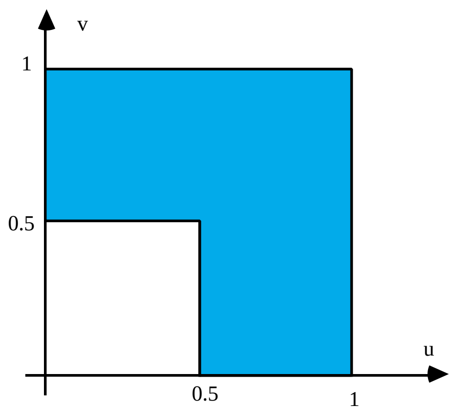
2. The random point ($\mathbf{X}$,$\mathbf{Y}$) is uniformly distributed on the shaded region shown in the left-hand figure below.

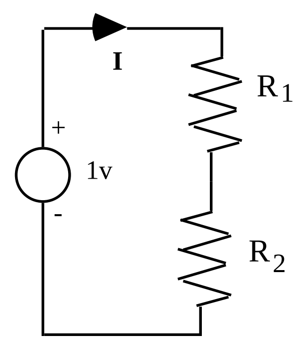
(a) Find the marginal pdf $f_{\mathbf{X}}(u)$ of the random variable $\mathbf{X}$.

(b) *Write down* the marginal pdf $f_{\mathbf{Y}}(v)$ of the random variable $\mathbf{Y}$ from your answer to part (b).

(c) Find $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\}$.

(d) What is $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$, the conditional pdf of $\mathbf{X}$ given that $\mathbf{Y} = \alpha$, if α satisfies $0 < \alpha < 1/2$?
What is $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$, the conditional pdf of $\mathbf{X}$ given that $\mathbf{Y} = \alpha$, if α satisfies $1/2 < \alpha < 1$?
Now, apply the theorem of total probability to compute the unconditional pdf of $\mathbf{X}$ from $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$. Do you get the same answer as in part (a)?





3. Two resistors are connected in series to a one-volt voltage source as shown in the right-hand diagram above. Suppose that the resistance values $\mathbf{R}_1$ and $\mathbf{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval (0, 1). Find the pdf $f_{\mathbf{I}}(a)$ of the current $\mathbf{I}$ (measured in amperes) in the circuit.

4. Let ($\mathbf{X}$, $\mathbf{Y}$) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$
Find the pdf of $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}$.

5. (Unbelievable but true: this problem is easier than it looks…).

(a) If $\mathbf{X}$ is $N(0,\sigma^2)$, use the magic formula in Example 7b, Chapter 5.7 of Ross to show that $\mathbf{X}^2$ has gamma pdf with parameter $(1/2,1/2\sigma^2)$.

(b) Now, suppose that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are independent $N(0,\sigma^2)$ random variables. Then $\mathbf{X}^2$, $\mathbf{Y}^2$, and $\mathbf{Z}^2$ are independent gamma random variables with parameter $(1/2,1/2\sigma^2)$. Use the comment immediately following the proof of Proposition 3.1 (p. 267, 5th ed. or p. 262, 6th ed.) of Ross to *state* what the *type* of pdf of $\mathbf{W} = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$ is, and write down *explicitly* the exact pdf. What *is* the numerical value of $f_{\mathbf{W}}(5)$ if $\sigma^2 = 4$?

(c) Prove that $E[\mathbf{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer instead of using LOTUS, shame on you!

(d) In a physical application, **X**, **Y**, and **Z** represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $\mathbf{H} = (1/2)m\mathbf{W}$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the average kinetic energy is $E[\mathbf{H}] = E[(1/2)m\mathbf{W}] = (1/2)mE[\mathbf{W}] = (3/2)m\sigma^2 = (3/2)kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in °K. (Note that the average energy is (1/2)kT per dimension.) Show that the kinetic energy **H** has the Maxwell-Boltzmann pdf $f_{\mathbf{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-3/2}\sqrt{\beta}\cdot\exp(-\beta/kT)$, $\beta > 0$.

(e) $\mathbf{V} = \sqrt{\mathbf{W}} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}$ is the "speed" of the molecule. Show that the pdf of **V** is $f_{\mathbf{V}}(\gamma) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right)$, $\gamma > 0$ cf. Theoretical Exercise 1 of Chapter 5.

(f) What is the average speed of the molecule?

6. The number of hours **R** that a student spends **r**eading about probability in preparation for the ECE 313 Final Examination and the number of hours **S** that the student spends **s**leeping can be modeled as random variables with joint probability density function

$$f_{\mathbf{R},\mathbf{S}}(x,y) = \begin{cases} K, & 10 \le x + y \le 20,\ x \ge 0,\ y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of K?

(b) What is the marginal pdf of **R**?

(c) Unfortunately, the more the student tries to read about probability, the more confused the student gets. Also, the less the student sleeps, the more tired the student gets. As a result, the student's *percentage* score **T** on the Final Exam is related to **S** and **R** via the equation

$$\mathbf{T} = 50 + 2.5(\mathbf{S} - \mathbf{R}).$$

Find the pdf of **T**.

(d) **Noncredit exercise:** Should **S** have denoted **s**tudying and **R** denoted **r**esting instead?