Assigned: Wednesday, April 3, 2002
Due: Wednesday, April 10, 2002

Reminder: **Hour Exam II** is scheduled for **Monday April 8, 7:00 pm to 8:00 pm,** in **Room 119 Materials Science and Engineering Building**

One 8.5" by 11" sheet of notes is allowed.

Calculators, laptop computers, Palm Pilots, pagers, SMS, etc are not allowed.

The material covered on Problem Sets 5-10 is included on the exam. Lecture material is Lectures 18-29, but decision-making involving continuous random variables will not be tested on Hour Exam II. Also, discrete random variables treated earlier in the course re-appeared in Poisson random processes, DeMoivre-LaPlace theorem etc., so don't neglect them entirely. Coverage of material from Ross is as follows:

Chapter 3 especially the material on independent events

Chapter 4: you should be able to use discrete random variables when needed
Chapter 5: all material except Sections 5.6.2 and 5.6.4

Chapter 8.2 (Chebyshev Inequality) and 9.1 (Poisson Process)
You should have the pdfs/pmfs, means, variances etc. of binomial, Poisson, geometric, negative binomial; uniform, exponential, gamma, and Gaussian random variables on your sheet of notes

Additional material (not always in Ross but covered in class and on homework) can be found in the Powerpoint slides and my Lecture Notes.

Reading: Ross, Chapter 5 and Chapter 6
Noncredit Exercises: Ross, Chapter 5: Problems 15-38; Chapter 6: Problems 1, 8-15, 20-23
Problems:

1. [Read Example 3d on pp. 198-199 first.] Let the (straight) line segment ACB be a diameter of a circle of unit radius and center C. Consider an arc AD of the circle where the length **X** of the *arc* (measured clockwise around the circle) is a random variable uniformly distributed on $[0,2\pi)$. Now consider the "random chord" AD.

(a) Find the probability that the length **L** of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

(b) Express **L** as a function of the random variable **X**, and find the probability density function for **L**.

2. ["Give me an A! Give me a D! Give me a converter! What have you got? An A/D converter! Go Team!"]
A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where $\mathbf{Y} = \alpha$ if $\mathbf{X} > 0$ and $\mathbf{Y} = -\alpha$ if $\mathbf{X} \le 0$) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as $\mathbf{Z} = (\mathbf{X}-\mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X}-\alpha)^2 & \text{if } \mathbf{X} > 0 \\ (\mathbf{X}+\alpha)^2 & \text{if } \mathbf{X} \le 0. \end{cases}$

So we want to choose α so that $E[\mathbf{Z}]$ is as small as possible. Use LOTUS to e-zily find $E[\mathbf{Z}]$ as a function of α, and then find the value of α that minimizes $E[\mathbf{Z}]$.

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, $\mathbf{W} = 3$ if $\mathbf{X} \ge 2.5$, $\mathbf{W} = 2$ if $1.5 \le \mathbf{X} < 2.5$, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is ($\mathbf{Z}_2$, $\mathbf{Z}_1$, $\mathbf{Z}_0$). What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

(f) **Noncredit exercise (but a real-life engineering problem!):** Suppose that **W** takes on values $-3\alpha, -2\alpha, -\alpha, 0, +\alpha, +2\alpha, +3\alpha$ and quantization is as before: **X** is mapped to the nearest **W** value. What value of α minimizes $E[(\mathbf{X} - \mathbf{W})^2]$?

3. The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable **X** with pdf $f(u) = (bu)\bullet\exp(-bu^2/2)$ for $u > 0$ (Ross, p. 216). The system fails at time t, i.e. $\mathbf{X} = t$ is observed to have occurred on this trial. What is the maximum-likelihood estimate of the parameter b that occurs in the pdf and hazard rate? Remember that the maximum-likelihood estimate $\hat{b}$ of the parameter b maximizes the pdf at the observed value t. Thus, for given t, what value of b maximizes $(bt)\exp(-bt^2/2)$?

4. If hypothesis H_0 is true, the pdf of **X** is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of **X** is exponential with parameter 10.

(a) Sketch the two pdfs.

(b) State the maximum-likelihood decision rule in terms of a threshold test on the observed value u of the random variable **X** instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ with 1.

(c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part (b)?

(d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to (π_0/π_1). Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable **X**.

(e) If $\pi_0 = \pi_1/2$, what is the average probability of error of the Bayesian decision rule?

(f) What is the average error probability of a decision rule that always decides H_0 is the true hypothesis, regardless of the value taken on by **X**?

(g) Show that if $\pi_0 > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by **X**.

5. The random variable **X** models a physical parameter. If hypothesis H_0 is true, then, $f_0(u)$, the pdf of **X**, is Gaussian with mean 0 and variance a^2. On the other hand, if hypothesis H_1 is true, then $f_1(u)$, the pdf of **X**, is Gaussian with mean 0 and variance $b^2 > a^2$.

(a) Suppose that H_0 and H_1 have equal probability. Thus, for i = 0, 1, the pdf of **X** when hypothesis H_i is true can be thought of as the *conditional* pdf of **X** given that H_i occurred, i.e. $f_{\mathbf{X}|H_i}(u|H_i)$. Write an expression for the *unconditional* pdf of **X**. Is the unconditional pdf of **X** a Gaussian pdf?

(b) What is the likelihood ratio? Simplify your answer.

(c) What is the maximum-likelihood decision rule, and what are the false alarm probability and the missed detection probability of this rule?

6. The discrete random variables **X** and **Y** have joint pmf $p_{\mathbf{X},\mathbf{Y}}(u,v)$ given by

4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
–1	1/12	1/6	1/12	0
v↑ / u→	0	1	3	5

(a) Find the marginal pmfs $p_{\mathbf{X}}(u)$ and $p_{\mathbf{Y}}(v)$ of **X** and **Y**.

(b) Are the random variables **X** and **Y** independent ?

(c) Find $P\{\mathbf{X} \leq \mathbf{Y}\}$ and $P\{\mathbf{X} + \mathbf{Y} \leq 8\}$.

(d) Find $p_{\mathbf{X}|\mathbf{Y}}(u|3)$, $E[\mathbf{X}|\mathbf{Y}=3]$, and $\mathrm{var}(\mathbf{X}|\mathbf{Y}=3)$.