

Assigned: Wednesday, March 27, 2002

Due: Wednesday, April 3, 2002

Reading: Ross, Chapter 5, Chapter 9.1

Noncredit Exercises: Ross, Chapter 5: Problems 11, 13, 15-38; Theoretical Exercises 12-16, 21, 26, 28-30

Problems:

1. Let X denote a unit Gaussian random variable with pdf $f_X(u)$ and CDF $F_X(x)$.

(a) What is the derivative of $\exp(-u^2/2)$ with respect to u ? Use this result to find $E[|X|]$.

$$\text{Now, let } Q(x) = \int_x^\infty f_X(u) du = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du = 1 - F_X(x).$$

(b) Some tables list the values of $Q(x)$ (instead of $F_X(x)$) for large values of x . Why might the tabulator have chosen to specify $Q(x)$ instead of $F_X(x)$? Explain briefly.

The rest of this problem leads you through a derivation of the bounds $(1/x - 1/x^3) < Q(x) < (x)/x$ mentioned in class and it also asks you to prove another simpler bound on $Q(x)$.

(c) Write the integrand for $Q(x)$ as $(\sqrt{2\pi})^{-1} u^{-1} (u \exp(-u^2/2))$ and integrate by parts to deduce that $Q(x) < x^{-1} \exp(-x^2/2)$ for $x > 0$. Repeat the trick of re-writing and integrating by parts to show that $x^{-1} \exp(-x^2/2) - x^{-3} \exp(-x^2/2) < Q(x)$. Are these bounds useful as $x \rightarrow 0$? Why or why not? What is the asymptotic value of the ratio of the bounds as $x \rightarrow 0$?

(d) A useful bound when x is small is $Q(x) \approx (1/2)\exp(-x^2/2)$ for $x \rightarrow 0$ in which equality holds only at $x = 0$. Derive this bound by first showing that $t^2 - x^2 > (t - x)^2$ for $t > x > 0$ and

$$\text{then applying this result to } \exp(x^2/2)Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{t^2 - x^2}{2}\right) dt$$

(e) For what values of x is this smaller than the upper bound of Eq.(4.4)?

2. Do **either** part (a) **or** part (b). Then do parts (c)–(e).

(a) Attach to your homework a **photocopy** of your calculator's manual page(s) that explains which **formula** your calculator uses to compute $Q(x)$. Reading the page might help too! Note: I **do not want** to know **which buttons** you have to press in order to find $Q(x)$; I want to know **what formula** your calculator uses internally to find $Q(x)$.

The xerographically-challenged are permitted to just copy the relevant formulas to their homework. **NEXT:** press the appropriate buttons to **find Q(5)**.

If your calculator cannot compute $Q(x)$, or if the manual does not state what formula is used to calculate $Q(x)$ but just tells you which buttons to press, or if you have lost the manual, do part (b) instead.

(b) Read Chapter 26.2 of Abramowitz and Stegun (*reference book (not a reserve book)* in Grainger Engineering Library), and use Equation 26.2.17 to **calculate Q(5)**.

This formula is also given on Slide 30 of Lecture 26 in the Powerpoint slides.

(c) The number found in part (a) or (b) is just an *approximation* to the value of $Q(5)$. Use the maximum error specification to find the *range* in which the actual value of $Q(5)$ must necessarily lie. What is the *maximum relative error* in the approximation to $Q(5)$ that you

found in part (a) or (b)? Note: the relative error is defined as $\frac{|\text{true value} - \text{computed value}|}{\text{true value}}$

expressed as a percentage.

(d) On p. 972, Abramowitz and Stegun give the value of $-\log_{10}Q(5)$ to be 6.54265.... Blindly trust your calculator to do the exponentiation correctly and find the *actual relative error* in the approximation to $Q(5)$ that you found in part (a) or (b). What would the actual relative

- error have been if you had simply used the upper bound $(5)/5$ as an approximation to $Q(5)$? What if you had used the lower bound $(1/5 - 1/5^3)$ (5) as an approximation to $Q(5)$?
- (e) Explain why the “much easier” Equation 26.2.18 of Abramowitz and Stegun is not particularly useful for computing $Q(5)$.
3. A signal $x(t) = \exp(-t^2)$, $-\infty < t < \infty$, is passed through a low-pass filter whose transfer function is $H(f) = \begin{cases} 2, & -1 < f < 1, \\ 0, & 1 < |f| < \infty. \end{cases}$ Let $y(t)$ denote the output of the filter. Compute the value of $y(0)$. A numerical answer is desired. [Hint: $X(f) = \exp(-f^2)$, $-\infty < f < \infty$.]
4. \mathbf{X} is a continuous random variable with pdf $f_{\mathbf{X}}(u) = 0.5 \exp(-|u|)$, $-\infty < u < \infty$.
- (a) What is the value of $P\{\mathbf{X} > \ln 2\}$?
- (b) Find the conditional probability that $P\{|\mathbf{X}| > \ln 2\}$ given that $\{\mathbf{X} > \ln 2\}$.
- (c) Find the numerical value of $P\{\cos(\mathbf{X}/2) < 0\}$.
- (d) Now suppose that \mathbf{X} denotes the voltage applied to a semiconductor diode, and that the current \mathbf{Y} is given by $\mathbf{Y} = e^{\mathbf{X}} - 1$. Find the pdf of \mathbf{Y} .
5. The random variable \mathbf{X} has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$
- Let $\mathbf{Y} = (1 - \mathbf{X})^2$.
- (a) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable \mathbf{Y} ? Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v , $-\infty < v < \infty$.
- (b) Show that the $F_{\mathbf{Y}}(v)$ that you found in part (a) is a nondecreasing continuous function.
6. The radius of a sphere is a random variable \mathbf{R} with pdf $f_{\mathbf{R}}(r) = \begin{cases} 3r^2, & 0 < r < 1, \\ 0, & \text{elsewhere.} \end{cases}$
- (a) Use LOTUS to find the average radius, average volume and average surface area of the sphere. Does a sphere of average radius also have average volume? Does a sphere of average radius also have average surface area?
- (b) Find the CDF $F_{\mathbf{V}}(v)$ and pdf $f_{\mathbf{V}}(v)$ of \mathbf{V} , the volume of the sphere.
- (c) Find $E[\mathbf{V}]$ directly from this pdf. Do you get the same answer as in part (a)? Why not?
- (d) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density \mathbf{S} on the sphere?
7. Ross, Problem 35, p. 231 (6th edition) or Problem 33, p. 236 (5th edition)