Assigned: Wednesday, February 27, 2002
Due: Wednesday, March 6, 2002
Reading: Ross, Chapter 3
Noncredit Exercises: Ross pp. 104-117: 53, 58, 59, 62, 63, 70-74, 78, 81

Reminder: **Hour Exam I** is scheduled for **Monday March 4, 7:00 pm to 8:00 pm,** in **Room 119 Materials Science and Engineering Building**

One 8.5" by 11" sheet of notes is allowed.

Calculators, laptop computers, Palm Pilots, pagers, SMS, etc are not allowed.

The material covered on Problem Sets 1-6 is included on the exam. *This* Problem Set also has material on decision theory, and thus, working these problems will help you prepare for the exam. Coverage of material from Ross is as follows:

Chapter 1 (except Section 1.6)

Chapter 2 (except Section 2.6)

Chapter 3 (except for Section 3.4 on independent events. However, note that we considered Example 4f on pp. 86-87 in the context of random variables)

Chapter 4.1–4.8.2. Material on hypergeometric and Zipf random variables is not included, nor is the material on the CDF (Section 4.9)

However, you must know (or have on your sheet of notes) the pmfs, means, variances etc. of the binomial, Poisson, geometric, and negative binomial random variables.

Additional material (not always in Ross but covered in class and on homework) can be found in the Powerpoint slides and my Lecture Notes.

Problems:

1. ["…From the town of Bedrock, They're a page right out of history…"] Fred suggests that he and Wilma play a game in which they will take turns tossing a fair coin; the first one to toss a Head wins. Fred proposes that he will toss first. Wilma agrees to this, but, having taken ECE 313, she knows that she is at an disadvantage. So, she demands that in succeeding games, the *loser* of the previous game gets to toss first. For $n \ge 1$, let p_n and $q_n = 1 - p_n$ respectively denote the probabilities that Fred and Wilma win the n-th game. We saw in class that $p_1 = 2/3 > q_1 = 1/3$,

(a) Use the theorem of total probability to show that $p_2 = 4/9 < q_2 = 5/9$. More generally, show that for $n \ge 2$, $p_n = (2/3) - (1/3)p_{n-1}$ and $q_n = (2/3) - (1/3)q_{n-1}$ and use these difference equations to find p_n and q_n in terms of p_1 and q_1.

(If you never learned in Math 285/286 or Math 315 (or ECE 310) how to solve difference equations, assume that **for all values of n,** p_n can be expressed as $a + b\alpha^n$. Substitute into the above difference equation and solve for α and a; the value of b is obtained from the "initial condition" $p_1 = 2/3$. Repeat for q_n — it is the same difference equation but the "initial condition" $q_1 = 1/3$ is different.)

(b) What is the limit as $n \to \infty$ of p_n and q_n ? Is this game asymptotically fair?

(c) Fred now proposes that instead of the first one to toss a head winning the game, the first one who *matches* the previous toss (whether the previous toss is part of the current game or the last toss of the previous game) wins. Wilma accepts but generously insists that, as before, Fred still toss first (so that the poor schmuck has no previous toss to match on his first toss!). What are p_1 and q_1 now? p_2 and q_2? p_3 and q_3? Is this game asymptotically fair? Assume as before that the loser of each game tosses first in the next game.

2. ["Reach out and touch someone…"] The probability that you can hear the sound of a pin dropping onto a table during a long-distance telephone call from Urbana to Hollywood is p_0 if the call is being carried over the AT&T network and p_1 if the call is being carried over the Sprint network. Assume that both p_0 and p_1 are quite small and that $p_1 > p_0$. You call Miss Candice Bergen from a payphone owned by Sleazo Telecom Corporation which happens to lease its long-distance lines either from AT&T or Sprint (but you don't know which!), and she very graciously agrees to drop pins one by one onto a table until you hear the sound of one dropping. Thus, if the first pin that you hear is the **X**th, then **X** is a geometric random variable with parameter p_0 or p_1 accordingly as the call is being carried by AT&T or Sprint. Suppose you hear Miss Bergen counting “One, two, three, … ” as she drops the pins, but you don’t hear the sound made by the pins as they land on the table until she says “thirtyfour” and you finally hear the sound of the pin dropping onto the table.

(a) Let H_0 and H_1 denote the hypotheses that the call is being carried by AT&T and Sprint respectively. What is the likelihood ratio LR for the observation that $\mathbf{X} = 34$?

(b) The maximum-likelihood decision compares LR to the threshold 1 and announces in favor of H_0 and H_1 according as LR < 1 or LR > 1 . Show that this decision rule can be expressed in terms of a threshold test on the *observed value* of **X**.

(c) If $p_1 = 0.04$ and $p_0 = 0.02$, what is the maximum-likelihood decision when $\mathbf{X} = 34$?

(d) AT&T is the lessor of 95% of all long-distance telephone lines while Sprint is the lessor of the remaining 5%, and thus it is reasonable to assume that $P(H_0) = 0.95$. What is the Bayesian (minimum-error-probability) decision when $\mathbf{X} = 34$?

(e) Noncredit exercise: Whom do you think is *my* long-distance carrier?

3. [“I’m leaving on a prop plane”] Consider again Problem #3 of Problem Set #5 in which 15 of the 105 passengers who hold reservations are arriving in Chicago on a connecting flight. If the connecting flight is on time, all 15 show up for the flight to Champaign; else, none of the 15 shows up. Let **Y** denote the number of *nonconnecting* passengers who actually show up for the flight. Let H_0 denote the hypothesis that the connecting flight is late, and H_1 the hypothesis that the connecting flight is on time. It is reasonable to assume that the pmf of **Y** is the same regardless of which hypothesis is true, and hence we model **Y** as a binomial random variable with parameters (90, 0.9). On the other hand, **X**, the *total* number of passengers showing up for the flight, equals **Y** if H_0 is true, while if H_1 is true, then $\mathbf{X} = 15 + \mathbf{Y}$, and thus the pmf of **X** *does* depend on which hypothesis is true.

(a) Suppose that the gate agent observes that $\mathbf{X} = 86$. What is $P\{\mathbf{X} = 86\}$ when H_0 is the true hypothesis? What is $P\{\mathbf{X} = 86\}$ when H_1 is the true hypothesis? What is the value of the likelihood ratio when $\mathbf{X} = 86$, and what is the agent’s maximum-likelihood decision as to whether the connecting flight is late?

(b) Repeat part (a) for the case when the gate agent observes that $\mathbf{X} = 96$.

(c) The gate agent knows that $\pi_0 = P\{H_0 \text{ is the true hypothesis}\} = 2/3$. For each of the two observations considered in parts (a) and (b), what is the agent’s MAP (or Bayesian or minimum-probability-of-error} decision as to whether the connecting flight is late?

(d) What is the probability that all passengers who show up get a seat? Given that all passengers who showed up got a seat, find the (conditional) probability that the connecting flight was late.

4. [“It a’in’t about bipartisan politics; it’s about …”] The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides *independently* of the other groups whether to support or oppose the motion. *All* members of the group then vote in accordance with the caucus decision.

For those who think that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a bargain price…

(a) Let A, B, C, and D respectively denote the events that the four groups vote for a spending plan that will lead to 50% increase in the DoD budget over the next two years. Suppose that the probabilities of these independent events are P(A) = 0.9, P(B) = 0.6 P(C) = 0.5 and P(D) = 0.2. What is the probability that the bill passes?

(b) The President vetoes the bill as a budget-breaker. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities P(E) = 0.99, P(F) = 0.4, P(G) = 0.6, and P(H) = 0.1, what is the probability that the motion to override the veto passes ?

Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

5. [“Tennis, anyone?”] Consider the following simplified model for a game of tennis. On each serve, let p denote the probability that player A wins the point, and q = 1–p the probability that player B wins the point. Assume that the outcome of each serve is independent of all others. Player A wins the game if the score reaches 4–0, 4–1, or 4–2, while B wins the game if the score reaches 2–4, 1–4, or 0–4. Else, the score reaches 3–3 (called deuce) and from this point onwards, the game continues until one player is two points ahead of the other, and thereby wins the game.

(a) Find the probabilities that the score reaches 4–0, 4–1, or 4–2 and the probabilities that the score reaches 2–4, 1–4, or 0–4. I need 6 answers here!

(b) Find P(score reaches deuce). Show that the sum of the seven probabilities obtained in parts (a) and (b) is 1 regardless of the value of p.

(c) Given that the score is deuce, what is P(A wins the next two points)? (This means A wins the game). What is P(B wins the next two points)? (This means B wins the game). What is the probability that both players win one point each? In this case, the score is tied again, and is also called deuce.

(d) Once the score reaches deuce, there *may* be further deuces until ultimately, either A or B wins both points and thereby wins the game. What is the probability that A ultimately wins the game given that the score is deuce? What is the probability that B ultimately wins the game given that the score is deuce? (Hint: these answers are different from those of part (c)) What is the probability that the game goes on forever with the score continuing to reach deuce after every two points?

(e) Use the results of parts (a)–(d) to express the probability that A wins the game as a function f(p) of p. A little thought shows that B wins with probability f(q) = f(1–p). Now, if p = 0, A wins no points which makes it difficult for him to win any games. Does your function f(p) satisfy f(0) = 0? If not, what does your f(p) give as the probability that A wins a game while losing every point? Similarly, if A wins every point, he is sure to win the game. Does your function f(p) satisfy f(1) = 1? If not, what does your f(p) give as the probability that A loses a game while winning every point? Other reasonable properties of f(p) are f(0.5) = 0.5, f(p) + f(1–p) =1. Which of these is satisfied by your function f(p)?

(f) Expand f(p) in a Taylor series in the neighborhood of p = 0.5 (only the first two terms are needed) What does this say about the probability of winning a game if $p = 0.5 + \varepsilon$ where ε is very small?

(g) Use your favorite graphing program to sketch f(p) as a function of p for $0 \leq p \leq 1$. Determine the minimum value of p for which $f(p) \geq 2/3$.

6. ["Baby needs new shoes!] The dice game of craps (see Ross, p. 58 in 5th ed. or p. 56 in 6th ed.) begins with the player (called the shooter) rolling two fair dice. If the result is a 2, or 3, or 12, the shooter loses, while if the result is a 7 or 11, the shooter wins. The shooter who rolls any of 4, 5, 6, 8, 9, 10 has neither won nor lost (as yet). What happens then is discussed in part (b).

(a) What is the probability that the shooter loses on the first roll? What is the probability that the shooter wins on the first roll?

(b) If the sum of the dice on the **first roll** is any of 4, 5, 6, 8, 9, 10, that number is called the **shooter's point.** For **each** number i in the set {4, 5, 6, 8, 9, 10}, find the probability that the shooter's point is i. I need six answers here, folks!

(c) Suppose that the shooter's point is i where i is some number in {4, 5, 6, 8, 9, 10}. The shooter now rolls the two dice again. If the result is a 7, the shooter loses (craps out.)
If the result is i, the shooter wins (this is referred to as making the point). If the result is neither i nor 7, the shooter rolls again. This process continues until the shooter either makes the point or craps out. Given that the shooter's point is i, what is the conditional probability that the shooter makes the point? Naturally, the answer depends on i, so here too, I need six answers.

(d) Use the above results to compute the probability of winning at craps.

(e) Given that the shooter's point is 8, what is the probability that the shooter makes it "the hard way," that is, by rolling two fours? Generally, bets are offered at 10-to-1 odds that the shooter makes the point 8 the hard way. That is, if you bet $1, you win $10 (plus your $1 back!) if the shooter makes 8 the hard way; and you lose the $1 that you bet if the shooter craps out or makes 8 by rolling 2-6, 3-5, 5-3, or 6-2). In the long run over many such bets, do you expect to make money, or lose money, or come out even?